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Generalized Hyers-Ulam Stability of a Cubic reciprocal Functional Equation

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Authors' contributions

This work was carried out in collaboration between both the authors. Author KR designed the study, wrote the first draft of the manuscript and managed literature searches. Author SS managed the analyses of the study and literature searches. Both the authors read and approved the final manuscript.

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Abstract

In this paper, we obtain the generalized Hyers-Ulam stability of a new cubic reciprocal functional equation of the form

$$f(2x+y)+f(2x-y)=\frac{4f(x)f(y)\left[4f(y)+3f(x)^{\frac{2}{3}}f(y)^{\frac{1}{3}}\right]}{\left(4f(y)^{\frac{2}{3}}-f(x)^{\frac{2}{3}}\right)^{3}}$$

in non-Archimedean fields with $f(y)\neq \frac{1}{8}f(x)$. We also extend the results related to Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of powers of norms for the same equation.

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1 Introduction

The classical theory of stability of functional equations is instigated by the question of S.M. Ulam [1] in the year 1940. In the subsequent year, D.H. Hyers [2] was the foremost mathematician to establish the result connected with the stability of functional equations. The result obtained by D.H. Hyers is called as Hyers-Ulam stability of functional equation. Later in the year 1950, T. Aoki [3] made further simplification to the theorem of D.H. Hyers. In the year 1978, Th.M. Rassias [4] took a broad view in the Hyers' result by taking the upper bound as sum of powers of norms. The theorem proved by Th.M. Rassias persuaded a lot of mathematicians to work on the stability of various functional equations and the result obtained by Th.M. Rassias is recognized as Hyers-Ulam-Rassias stability of functional equation. J.M. Rassias [5] provided further generalization of the result of D.H. Hyers by using weaker conditions controlled by a product of different powers of norms. The result proved by J.M. Rassias is termed as Ulam-Gavruta-Rassias stability of functional equation. Further, in the year 1994, P. Gavruta [6] provided a generalization of Th.M. Rassias' theorem by replacing a general control function as upper bound. The stability result ascertained by P. Gavruta is celebrated as generalized Hyers-Ulam stability of functional equation. In the year 2008, J.M. Rassias et al. [7] investigated the stability of quadratic functional equation

$$f(mx+y)+f(mx-y)=2f(x+y)+2f(x-y)+2\left(m^2-2\right)f(x)-2f(y)$$

for any arbitrary but fixed real constant m with $m \neq 0$; $m \neq \pm 1$; $m \neq \pm\sqrt{2}$ using mixed product-sum of powers of norms. This stability result acquired by J.M. Rassias is known as J.M. Rassias stability involving mixed product-sum of powers of norms.

In the year 2010, K. Ravi and B.V. Senthil Kumar [8] proved Ulam-Gavruta-Rassias stability for a new reciprocal type functional equation

$$f(x+y)=\frac{f(x)f(y)}{f(x)+f(y)} \tag{1.1}$$

where $f : X \to \mathbb{R}$ is a mapping with X as the space of non-zero real numbers. The reciprocal function $g(x) = \frac{c}{x}$ is a solution of the functional equation (1.1).

The other results pertaining to stability of different reciprocal type functional equations can be found in ([9], [10], [11]).

A. Bodaghi and S.O. Kim [12] introduced and studied the Ulam-Gavruta-Rassias stability for the quadratic reciprocal functional equation

$$f(2x+y)+f(2x-y)=\frac{2f(x)f(y)[4f(y)+f(x)]}{(4f(y)-f(x))^2}. \tag{1.2}$$

The quadratic reciprocal function $f(x) = \frac{c}{x^2}$ is a solution of the functional equation (1.2).

A. Bodaghi and Y. Ebrahimdoost [13] generalized the equation (1.2) as

$$f((a+1)x+ay)+f((a+1)x-ay)=\frac{2f(x)f(y)\left[(a+1)^2f(y)+a^2f(x)\right]}{\left((a+1)^2f(y)-a^2f(x)\right)^2} \tag{1.3}$$

where $a \in \mathbb{Z}$ with $a \neq 0$ and established the generalized Hyers-Ulam-Rassias stability for the functional equation (1.3) in non-Archimedean fields.

K. Ravi et al. [14] investigated the generalized Hyers-Ulam-Rassias stability of a reciprocal-quadratic functional equation of the form

$$r(x+2y)+r(2x+y)=\frac{r(x)r(y)\left[5r(x)+5r(y)+8\sqrt{r(x)r(y)}\right]}{\left[2r(x)+2r(y)+5\sqrt{r(x)r(y)}\right]^2} \tag{1.4}$$

in intuitionistic fuzzy normed spaces.

In this paper, we obtain the generalized Hyers-Ulam stability of a cubic reciprocal functional equation of the form

$$f(2x+y)+f(2x-y)=\frac{4f(x)f(y)\left[4f(y)+3f(x)^{\frac{2}{3}}f(y)^{\frac{1}{3}}\right]}{\left(4f(y)^{\frac{2}{3}}-f(x)^{\frac{2}{3}}\right)^3} \tag{1.5}$$

in non-Archimedean fields. We extend the results concerning Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of powers of norms for the equation (1.5). It is easy to see that the reciprocal cubic function $f(x)=\frac{1}{x^3}$ is a solution of the functional equation (1.5).

2 Preliminaries

In this section, we recall the basic concepts of non-Archimedean filed, non-Archimedean norm and non-Archimedean space.

Definition 2.1. By a non-Archimedean field, we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r|=0$ if and only if $r=0$, $|rs|=|r||s|$ and $|r+s|\leq max\{|r|,|s|\}$ for all $r,s\in K$.

Clearly $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|.|$. A function $||.||:X\to\mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

(i) $||x||=0$ if and only if $x=0$;

(ii) $||rx||=|r|||x|| \qquad (r\in\mathbb{K},x\in X)$;

(iii) the strong triangle inequality (ultrametric); namely,

$$||x+y||\leq \max\{||x||,||y||\} \qquad (x,y\in X).$$

Then $(X,||.||)$ is called a non-Archimedean space. Due to the fact that

$$\|x_n-x_m\|\leq \max\{\|x_{j+1}-x_j\|:m\leq j\leq n-1\} \qquad (n>m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space, we mean that every Cauchy sequence is convergent in the space.

An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking everything but 0 into 1 and $|0|=0$. This valuation is called trivial. Another example of a non-Archimedean valuation on a field $\mathbb{A}$ is the mapping

$$|r|=\begin{cases}0 & \text{if } r=0\\ \frac{1}{r} & \text{if } r>0\\ -\frac{1}{r} & \text{if } r<0\end{cases}$$

for any $r \in \mathbb{A}$.

Let p be a prime number. For any non-zero rational number $x = p^r \frac{m}{n}$ in which m and n are coprime to the prime number p. Consider the p-adic absolute value $|x|_p = p^{-r}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ which is denoted by $\mathbb{Q}_p$ is said to be the p-adic number field. Note that if $p > 2$, then $|2^n| = 1$ for all integers n.

3 Generalized Hyers-Ulam Stability of Equation (1.5)

In this section, we investigate the generalized Hyers-Ulam stability of equation (1.5) in non-Archimedean fields. We also establish the results pertaining to Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by product-sum of powers of norms.

Throughout this section, we consider that $\mathbb{A}$ and $\mathbb{B}$ is a non-Archimedean field and a complete non-Archimedean field, respectively. From now on, for a non-Archimdean field $\mathbb{A}$, we put $\mathbb{A}^* = A - \{0\}$. For the purpose of simplification, let us define the difference operator $D_f : \mathbb{A}^* \times \mathbb{A}^* \to \mathbb{B}$ by

$$D_f(x, y) = f(2x + y) + f(2x - y) - \frac{4f(x)f(y)\left[4f(y) + 3f(x)^{\frac{2}{3}}f(y)^{\frac{1}{3}}\right]}{\left(4f(y)^{\frac{2}{3}} - f(x)^{\frac{2}{3}}\right)^3}$$

for all $x, y \in \mathbb{A}^*$.

In the following results, we assume $f(x) \neq 0$ and $x + y \neq 0$ for all $x, y \in \mathbb{R}$.

For proving our main results, we necessitate the following definition.

Definition 3.1. A mapping $f : \mathbb{A}^* \to \mathbb{B}$ is called as reciprocal cubic mapping if f satisfies equation (1.5).

Theorem 3.1. *Let $\varphi : \mathbb{A}^* \times \mathbb{A}^* \to \mathbb{B}^*$ be a function such that*

$$\lim_{n\to\infty} \left|\frac{1}{27}\right|^n \varphi\left(\frac{x}{3^{n+1}}, \frac{y}{3^{n+1}}\right) = 0 \tag{3.1}$$

for all $x, y \in \mathbb{A}^*$. Suppose that $f : \mathbb{A}^* \to \mathbb{B}$ is a mapping satisfying the inequality

$$|D_f(x, y)| \leq \varphi(x, y) \tag{3.2}$$

for all $x, y \in \mathbb{A}^*$. Then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \to \mathbb{B}$ such that

$$|f(x) - R_c(x)| \leq \max\left\{\left|\frac{1}{27}\right|^j \varphi\left(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}\right) : j \in \mathbb{N} \cup \{0\}\right\} \tag{3.3}$$

for all $x \in \mathbb{A}^*$.

Proof. Plugging (x, y) by (x, x) in (3.2), we obtain

$$\left|f(3x) - \frac{1}{27}f(x)\right| \leq \varphi(x, x) \tag{3.4}$$

for all $x \in \mathbb{A}^*$. Now, substituting x by $\frac{x}{3}$ in (3.4), we find

$$\left|f(x) - \frac{1}{27}f\left(\frac{x}{3}\right)\right| \leq \varphi\left(\frac{x}{3}, \frac{x}{3}\right) \tag{3.5}$$

for all $x \in \mathbb{A}^*$. Replacing x by $\frac{x}{2^n}$ in (3.5) and multiplying by $\left|\frac{1}{27^n}\right|^n$, we have

$$\left|\frac{1}{27^n} f\left(\frac{x}{3^n}\right) - \frac{1}{27^{n+1}} f\left(\frac{x}{3^{n+1}}\right)\right| \leq \left|\frac{1}{27}\right|^n \varphi\left(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}\right) \tag{3.6}$$

for all $x \in \mathbb{A}^*$. Hence we find that the sequence $\left\{\frac{1}{27^n} f\left(\frac{x}{3^n}\right)\right\}$ is a Cauchy sequence by (3.1) and (3.6). Since $\mathbb{B}$ is complete, we can define a mapping R_c given by

$$R_c(x) = \lim_{n\to\infty} \frac{1}{27^n} f\left(\frac{x}{3^n}\right). \tag{3.7}$$

For each $x \in \mathbb{A}^*$ and non-negative integers n, we have

$$\begin{aligned}\left|\frac{1}{27^n} f\left(\frac{x}{3^n}\right) - f(x)\right| &= \left|\sum_{i=0}^{n-1} \left\{\frac{1}{27^{i+1}} f\left(\frac{x}{3^{i+1}}\right) - \frac{1}{27^i} f\left(\frac{x}{3^i}\right)\right\}\right| \\ &\leq \max\left\{\left|\frac{1}{27^{i+1}} f\left(\frac{x}{3^{i+1}}\right) - \frac{1}{27^i} f\left(\frac{x}{3^i}\right)\right| : 0 \leq i < n\right\} \\ &\leq \max\left\{\left|\frac{1}{27}\right|^i \varphi\left(\frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}\right) : 0 \leq i < n\right\}. \end{aligned} \tag{3.8}$$

The inequality (3.3) is true using (3.7) and letting $n \to \infty$ in the above inequality (3.8). Using (3.1), (3.2) and (3.7), we have for all $x, y \in \mathbb{A}^*$

$$\begin{aligned}|D_{R_c}(x,y)| &= \lim_{n\to\infty} \left|\frac{1}{27}\right|^n \left|D_f\left(\frac{x}{3^n}, \frac{y}{3^n}\right)\right| \\ &\leq \lim_{n\to\infty} \left|\frac{1}{27}\right|^n \varphi\left(\frac{x}{3^n}, \frac{y}{3^n}\right) = 0.\end{aligned}$$

Thus the mapping R_c satisfies (1.5) and hence it is cubic reciprocal mapping. In order to prove the uniqueness of R_c, let us consider $R'_C : \mathbb{A}^* \to \mathbb{B}$ be another cubic reciprocal mapping satisfying (3.3). Then we have

$$\begin{aligned}&\left|R_c(x) - R'_C(x)\right| \\ &= \lim_{m\to\infty} \left|\frac{1}{27}\right|^m \left|R_c\left(\frac{x}{3^m} x\right) - R'_c\left(\frac{x}{3^m}\right)\right| \\ &\leq \lim_{m\to\infty} \left|\frac{1}{27}\right|^m \max\left\{\left|R_c\left(\frac{x}{3^m}\right) - f\left(\frac{x}{3^m}\right)\right|, \left|f\left(\frac{x}{3^m}\right) - R'_c\left(\frac{x}{3^m}\right)\right|\right\} \\ &\leq \lim_{m\to\infty} \lim_{n\to\infty} \max\left\{\max\left\{\left|\frac{1}{27}\right|^{i+m} \varphi\left(\frac{x}{3^{i+m+1}}, \frac{x}{3^{i+m+1}}\right) : m \leq i \leq n+m\right\}\right\} \\ &= 0\end{aligned}$$

for all $x \in \mathbb{A}^*$, which shows that R_c is unique. Hence the proof is complete. □

Theorem 3.2. *Let $\varphi : \mathbb{A}^* \times \mathbb{A}^* \to \mathbb{B}^*$ be a function such that*

$$\lim_{n\to\infty} |27|^n \varphi(3^n x, 3^n y) = 0 \tag{3.9}$$

for all $x, y \in \mathbb{A}^*$. Suppose that $f : \mathbb{A}^* \to \mathbb{B}$ is a mapping satisfying the inequality (3.2) for all $x, y \in \mathbb{A}^*$. Then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \to \mathbb{B}$ such that

$$|f(x) - R_c(x)| \leq \max\left\{|27|^{i+1} \varphi\left(3^i x, 3^i x\right) : i \in \mathbb{N} \cup \{0\}\right\} \tag{3.10}$$

for all $x \in \mathbb{A}^*$.

Proof. Switching y to x in (3.2) and multiplying by $|27|$, we find

$$|27f(3x) - f(x)| \leq |27|\varphi(x,x) \tag{3.11}$$

for all $x \in \mathbb{A}^*$. Now, replacing x by $3^n x$ in (3.11) and multiplying by $|27|^n$, we obtain

$$\left|27^n f\left(3^n x\right) - 27^{n+1} f\left(3^{n+1}x\right)\right| \leq |27|^{n+1}\varphi\left(3^n x, 3^n x\right) \tag{3.12}$$

for all $x \in \mathbb{A}^*$. It is easy to see that the sequence $\{3^n f(3^n x)\}$ is a Cauchy sequence by letting $n \to \infty$ in (3.12) and using (3.9). Since $\mathbb{B}$ is complete, this Cauchy sequence converges to a mapping $R_c : \mathbb{A}^* \to \mathbb{B}$ defined by

$$R_c(x) = \lim_{n\to\infty} 27^n f\left(3^n x\right). \tag{3.13}$$

For each $x \in \mathbb{A}^*$ and non-negative integers n, we have

$$\begin{aligned} \left|27^n f\left(3^n x\right) - f(x)\right| &= \left|\sum_{i=0}^{n-1} 27^{i+1} f\left(3^{i+1}x\right) - 27^i f\left(3^i x\right)\right| \\ &\leq \max\left\{\left|27^{i+1} f\left(3^{i+1}x\right) - 27^i f\left(3^i x\right)\right| : 0 \leq i < n\right\} \\ &\leq \max\left\{|27|^{i+1}\varphi\left(3^i x, 3^i x\right) : 0 \leq i < n\right\}. \end{aligned} \tag{3.14}$$

Applying (3.13) and letting $n \to \infty$, we find that the inequality (3.10) holds. From (3.9), (3.2) and (3.13), we have for all $x, y \in \mathbb{A}^*$

$$\begin{aligned} \left|D_{R_c}(x,y)\right| &= \lim_{n\to\infty} |27|^n \left|D_f\left(3^n x, 3^n y\right)\right| \\ &\leq \lim_{n\to\infty} |27|^n \varphi\left(3^n x, 3^n y\right) = 0. \end{aligned}$$

Hence the mapping R_c satisfies (1.5) and it is cubic reciprocal mapping. Now, let $R'_C : \mathbb{A}^* \to \mathbb{B}$ be another cubic reciprocal mapping satisfying (1.5). Then we have

$$\begin{aligned} &\left|R_c(x) - R'_c(x)\right| \\ &\quad= \lim_{m\to\infty} |27|^m \left|R_c\left(3^m x\right) - R'_c\left(3^m x\right)\right| \\ &\quad\leq \lim_{m\to\infty} |27|^m \max\left\{\left|R_c\left(3^m x\right) - f\left(3^m x\right)\right|, \left|f\left(3^m x\right) - R'_c\left(3^m x\right)\right|\right\} \\ &\quad\leq \lim_{m\to\infty}\lim_{n\to\infty} \max\left\{\max\left\{|27|^{i+m+1}\varphi\left(3^{i+m}x, 3^{i+m}x\right) : m \leq i \leq n+m\right\}\right\} \\ &\quad= 0 \end{aligned}$$

for all $x \in \mathbb{A}^*$, which proves that R_c is unique. □

Theorem 3.3. *Let $f : \mathbb{A}^* \to \mathbb{B}$ be a mapping for which there exists a constant $\epsilon > 0$ (independent of x and y) such that the functional inequality*

$$|D_f(x,y)| \leq \frac{26\epsilon}{27} \tag{3.15}$$

holds for all $x, y \in \mathbb{A}^*$. Then the limit

$$R_c(x) = \lim_{n\to\infty} \frac{1}{27^n} f\left(\frac{x}{3^n}\right) \tag{3.16}$$

exists for all $x \in \mathbb{A}^*$, $n \in \mathbb{N}$ and $R_c : \mathbb{A}^* \to \mathbb{B}$ is the unique mapping satisfying the cubic reciprocal functional equation (1.5) such that

$$|f(x) - R_c(x)| \leq \epsilon \tag{3.17}$$

for all $x \in \mathbb{A}^*$.

Proof. Putting $y = x$ in (3.15), we get

$$\left| f(3x) - \frac{1}{27} f(x) \right| \leq \frac{26\epsilon}{27} \tag{3.18}$$

for all $x \in \mathbb{A}^*$. Now, substituting x by $\frac{x}{3}$ in (3.18), we find

$$\left| f(x) - \frac{1}{27} f\left(\frac{x}{3}\right) \right| \leq \frac{26\epsilon}{27} \tag{3.19}$$

for all $x \in \mathbb{A}^*$. Replacing x by $\frac{x}{3}$ in (3.19) and then dividing both sides by 3^3, we have

$$\left| \frac{1}{3^3} f\left(\frac{x}{3}\right) - \frac{1}{3^6} f\left(\frac{x}{3^2}\right) \right| \leq \frac{26\epsilon}{27^2} \tag{3.20}$$

for all $x \in \mathbb{A}^*$. It follows from (3.19) and (3.19) that

$$\left| f(x) - \frac{1}{3^6} f\left(\frac{x}{3^2}\right) \right| \leq \frac{26\epsilon}{27} \left(1 + \frac{1}{3^3}\right) \tag{3.21}$$

for all $x \in \mathbb{A}^*$. The above process can be repeated to obtain

$$\left| f(x) - \frac{1}{3^{3n}} f\left(\frac{x}{3^n}\right) \right| \leq \frac{26\epsilon}{27} \left(1 + \frac{1}{3^3} + \frac{1}{3^6} + ... + \frac{1}{3^{3(n-1)}}\right) \tag{3.22}$$

for all $x \in \mathbb{A}^*$ and all $n \in \mathbb{N}$. In order to prove the convergence of the sequence $\left\{\frac{1}{3^{3n}} f\left(\frac{x}{3^n}\right)\right\}$, we have if $n > k > 0$, then by (3.23)

$$\begin{aligned}
\left| \frac{1}{3^{3n}} f\left(\frac{x}{3^n}\right) - \frac{1}{3^{3k}} f\left(\frac{x}{3^k}\right) \right| &= \frac{1}{3^{3k}} \left| \frac{1}{3^{3(n-k)}} f\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^k}\right) \right| \\
&= \frac{1}{3^{3k}} \left| \frac{1}{3^{3(n-k)}} f\left(\frac{y}{3^{n-k}}\right) - f(y) \right| &(3.23) \\
&\leq \frac{1}{3^{3k}} \frac{26\epsilon}{27} \left(1 + \frac{1}{3^3} + \frac{1}{3^6} + ... + \frac{1}{3^{3(n-k-1)}}\right) &(3.24) \\
&\leq 3^{-3k}\epsilon &(3.25)
\end{aligned}$$

for all $x \in \mathbb{A}^*$ in which $y = \frac{x}{3^k}$. The above result shows that the mentioned sequence is a Cauchy sequence and thus limit (3.16) exists for all $x \in \mathbb{A}^*$. Taking that n tends to infinity in (3.21), we can see that inequality (3.17) holds for all $x \in \mathbb{A}^*$. Replacing (x, y) by $\left(\frac{x}{3^n}, \frac{y}{3^n}\right)$, respectively in (3.15) and dividing both sides by 3^{3n}, we deduce that

$$\frac{1}{3^{3n}} \left| D_f \left(3^n x, 3^n y\right) \right| \leq \frac{26\epsilon}{3^{3(n+1)}} \tag{3.26}$$

holds for all $x, y \in \mathbb{A}^*$. Taking that n tends to infinity in (3.26), we see that $R_c(x)$ satisfies (3.15) for all $x, y \in \mathbb{A}^*$. To prove that $R_c'(x)$ is a unique quadratic reciprocal functional satisfying (3.15) subject to (3.18), let us consider a $R_c'(x) : \mathbb{A}^* \to \mathbb{B}$ to be another quadratic reciprocal function which satisfies (3.15) and inequality (3.18). Clearly $R_c(x)$ and $R_c'(x)$ satisfy (3.18) and using (3.17), we get

$$\begin{aligned}
\left| R_c(x) - R_c'(x) \right| &= \lim_{n\to\infty} \frac{1}{3^{3n}} \left| R_c\left(\frac{x}{3^n}\right) - R_c'\left(\frac{x}{3^n}\right) \right| \\
&\leq \lim_{n\to\infty} \frac{1}{3^{3n}} \left\{ \left| R_c\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right) \right| + \left| f\left(\frac{x}{3^n}\right) - R_c'\left(\frac{x}{3^n}\right) \right| \right\} \\
&= 0
\end{aligned}$$

for all $x \in \mathbb{A}^*$, which proves that R_c is unique. □

Corollary 3.1. *Let $c_1 \geq 0$ and $p \neq -3$, be fixed constants. If $f : \mathbb{A}^* \to \mathbb{B}$ satisfies*

$$|D_f(x, y)| \leq c_1 \left(|x|^p + |y|^p\right)$$

for all $x, y \in \mathbb{A}^*$, then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \to \mathbb{B}$ satisfying (1.5) and

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{2c_1}{|3|^p|3|^{p+3}} |x|^p, & \text{for } p > -3 \\ 2c_1|3|^3|3|^{p+3} |x|^p, & \text{for } p < -3 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. Considering $\varphi(x, y) = c_1 (|x|^p + |y|^p)$, for all $x, y \in \mathbb{A}^*$ in Theorem 3.1 with $p > -3$ and in Theorem 3.2 with $p < -3$ and proceeding by similar arguments as in Theorems 3.1 and 3.2. □

Corollary 3.2. *Let $f : \mathbb{A}^* \to \mathbb{B}$ be a mapping and let there exist real numbers $\alpha, \beta : \rho = \alpha + \beta \neq -3$. Let there exists $c_2 \geq 0$ such that*

$$|D_f(x, y)| \leq c_2 |x|^\alpha |y|^\beta$$

for all $x, y \in \mathbb{A}^*$. Then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \to \mathbb{B}$ satisfying (1.5) and

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{c_2}{|3|^\rho|3|^{\rho+3}} |x|^\rho, & \text{for } \rho > -3 \\ c_2|3|^3|3|^{+\rho+3} |x|^\rho, & \text{for } \rho < -3 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. The required results are obtained by choosing $\varphi(x, y) = c_2 |x|^\alpha |y|^\beta$, for all $x, y \in \mathbb{A}^*$ in Theorem 3.1 with $\rho > -3$ and in Theorem 3.2 with $\rho < -3$, the proof of the corollary is complete. □

Corollary 3.3. *Let $c_3 \geq 0$ and $r \neq -3$ be real numbers, and $f : \mathbb{A}^* \to \mathbb{B}$ be a mapping satisfying the functional inequality*

$$|D_f(x, y)| \leq c_3 \left(|x|^{\frac{r}{2}} |y|^{\frac{r}{2}} + (|x|^r + |y|^r)\right)$$

for all $x, y \in \mathbb{A}^*$. Then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \to \mathbb{B}$ satisfying (1.5) and

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{3c_3}{|3|^r|3|^{r+3}} |x|^r, & \text{for } r > -3 \\ 3c_3|3|^3|3|^{r+3} |x|^r, & \text{for } r < -3 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. The proof follows immediately by taking $\varphi(x, y) = \left(|x|^{\frac{r}{2}} |y|^{\frac{r}{2}} + (|x|^r + |y|^r)\right)$ in Theorem 3.1 with $r > -3$ and in Theorem 3.2 with $r < -3$. □

4 Conclusion

This is the first attempt made to study stability results on the cubic type rational functional equation. In this paper we have investigated a new type of cubic rational functional equation in non-Archimedean fields and we arrived the stability results in the sense of Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability and the results are more attractive.

Competing Interests

Authors have declared that no competing interests exist.

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