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Robertson – Walker Metric in (2 + 1) – Dimensions: The 2 – D Coordinate Subspaces and Their Curvature

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we will first construct a Robertson – Walker like metric in (2 + 1) – dimensional space. The easiest way of doing this is to consider a 2-dimensional coordinate space as a space embedded in a 3-dimensional hypersurface. The curvature of each surface is determined using the spatial part of the Robertson – Walker like metric constructed. Our main goal is to find out if the Robertson – Walker like metric in (2 + 1) – dimensional space can be used as a prototype model to study Robertson – Walker in (3 + 1) dimensions since calculations involved in higher dimensions are tedious.

Keywords: Robertson – Walker metric; curvature; hypersurface; spacetime; Christoffel symbols; Riemann curvature; Gaussian curvature.

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1. Introduction

In mathematics, curvature is one of the strongly related concepts in geometry. Geometry as defined by [1], is a branch of mathematics concerned with questions of shape, size, relative position of figures and the properties of space. The study of geometry started as early as the 6^{th} century BC. It began to see elements of formal mathematical science emerging in Greek mathematics [2]. Also in general relativity, curvature is a very important concept and it can be described in tensorial terms. It is a mathematical quantity involving the second derivative of the metric which represents the essence of a curve space. The space is curved if the curvature does not vanish.

Geometry has evolved through many phases and into modern times, it has expanded into non – Euclidean geometry and manifolds describing spaces that lie beyond the normal range of human experience. While geometry has evolved significantly throughout the years, there are some general concepts that are fundamental. These include the concept of point, line, plane, distance, angle, surface and curve as well as the more advanced notions of topology and manifolds [3]. Geometry has applications in many fields, including art, architecture, physics as well as to other branches of mathematics [4]. In this study, we looked at some concepts of geometry, specifically curvature of surfaces and distance.

Furthermore, curvature is seen as the rate of change in direction of a curve with respect to distance along the curve. To measure the curvature of a surface at a point, Euler in 1760, looked at cross sections of the surface made by planes that contain the line perpendicular (or normal) to the surface at the point. Euler called the curvatures of these cross sections the normal curvatures of the surface at the point. If the curve is a section of a surface, that is, the curve formed by the intersection of a plane with the surface, then the curvature of the surface at any given point can be determined by appropriate sectioning planes. The most useful planes are two that both contain the normal to the surface at the point. One of these planes produces the section with the greatest curvature among all such sections and the other produces that with the least. These two planes define the two so – called principal directions on the surface at the point. These directions lie at right angles to one another. The curvatures in the principal directions are called the principal curvatures of the surface.

The mean curvature of the surface at the point is either the sum of the principal curvatures or half that sum. The Gaussian curvature is the product of the principal curvatures. Gaussian curvature was used for the study because the researchers considered an isometric space which is an intrinsic characteristic of the surface independent of the coordinate system used. This is referred to as the Gauss' Theorema Egregium.

In classical cosmology theory, the cosmological principle is very paramount, which states that, on large scale, the Universe is homogeneous and isotropic [5,6,7]. That means that the Universe looks the same from each point and in all directions. These do not automatically imply one another. For example, a Universe with a uniform magnetic field is homogeneous, as all points are the same, but it fails to be isotropic because direction along the field lines can be distinguished from those perpendicular to them.

According to [8], the assumption that the Universe is homogeneous and isotropic is the foundation for the majority of modern cosmological models which helps scientist to study the nature of the Universe. Our Universe is currently best described using the standard cosmological model of particle physics and its extensions which is the Friedmann Robertson Walker (FRW) model [9,10,11]. The Friedmann Robertson Walker (FRW) models are established on the basis of the assumption that the Universe is homogeneous and isotropic on a large scale [12]. This implies that when this homogeneity and isotropy or both are broken down then these models can no longer predict the nature and behaviour of the Universe.

In the study done by [13], the models of the Universe were analysed consistent with the observed isotropy, entropy, element abundances and with the existence of galaxies. They emphasized on the four important facts about the Universe, for which theoretical explanations are still being sought which includes the Universe being considered as homogeneous and isotropic on a large scale.

A lot of research have been done on the relationship between the sign of curvature and shape of the Universe. There are basically three possible shapes of the Universe which is related to the sign of curvature of the space. These are; a flat Universe (Euclidean or zero curvature), a spherical or closed Universe (positive curvature) and hyperbolic or open Universe (negative curvature).

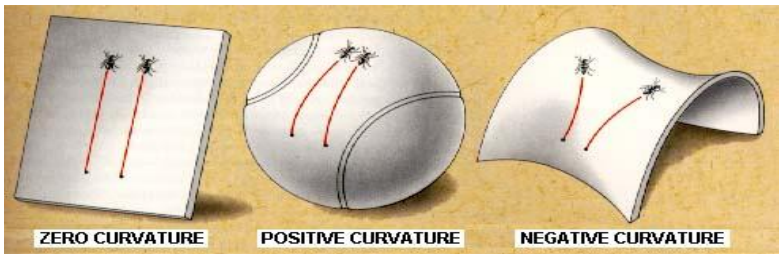

Fig. 1. Surfaces of zero curvature, positive curvature and negative curvature

In general relativity the metric tensor describes the local geometry of spacetime. Metric in general relates physical distances or intervals between events separated in space or time to the coordinates used to describe their position. In general relativity, four – dimensional spacetime in which the separation between space and time coordinates are not clear are used. However, it is possible to define a unique time coordinate called cosmic time and three spatial coordinates in a homogeneous and isotropic cosmology. The Robertson – Walker metric in general describes a curved space which is either expanding or contracting with cosmic time. It is named after the American mathematician and cosmologist, Howard Percy Robertson (1903 - 1961) and the English mathematician, Arthur Geoffrey Walker (1909 - 2001) [14].

In the standard model of cosmology, the homogeneous and isotropic universe is described by the Robertson – Walker metric, which in spherical coordinates has a line element denoted by [15,16,17,18,19];

$$ds^2 = -dt^2 + s^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left[d\theta^2 + \sin^2\theta d\phi^2\right]\right\}$$

where t and r, θ, ϕ are time and space spherical coordinates, and the signature of the metric is $(-,+,+,+)$, s(t) is an unknown function of time and k describes the curvature of the spatial section with values which is equals +1, 0 or -1.

In another scenario, [20] considered an observer (say, A) in an inertial frame who measures the density of stars and their velocities in the space around him. Because of the homogeneity and isotropy of the space, he would see the same mean density of stars (at one time, t) in the two different directions r and r', $\varphi_A(\vec{r}, t) = \varphi_A(\overrightarrow{r'}, t)$. Another observer (say, B) in another inertial frame, looking in the direction $\overrightarrow{r'}$ from his location would also see the same density of stars, $\varphi_B(\overrightarrow{r'}, t) = \varphi_A(\vec{r}, t)$. The velocity distribution of stars would also look the same to both observers, in fact in all directions, for instance in the $\overrightarrow{r'}$ direction, $V_B(\overrightarrow{r'}, t) = V_A(\overrightarrow{r'}, t)$. Hence, we conclude that the universe is homogeneous and isotropic.

When faced with a very difficult problem in mathematics, it is a good idea to start by solving an easier version of the same problem. So instead of thinking about the shape of our universe, we begin one dimension down, by considering how 2 – dimensional creatures living in a 2 – dimensional universe might think about possible shapes or curvatures for their universe [21].

According to [22], another miracle of the General Relativity models of the universe which cannot come out of the Newtonian model is the spatial curvature of the universe. We always think in terms of flat 3 – dimensional Euclidean geometry which is adequate for all terrestrial purposes. However, there are other geometries which satisfy the requirement that they should be same at all points in an isotropic universe.

If a two – dimensional space has curvature which is homogeneous and isotropic; its geometry can be specified by two quantities, k and S. The number k, called the curvature constant/parameter, k = 0 for a flat space, k = +1 for a positively curved space and k = -1 for a negatively curved space. If the space is curved, then the quantity S, which has dimensions of length, is the radius of curvature [23].

2. Main Thrust

First, the Robertson – Walker like metric in (2 + 1) – dimensional space is constructed. The easiest way to do this is to consider a 2 – dimensional coordinate space as a space embedded in a 3 – dimensional "hypersurface". For a space of constant negative curvature, the hypersurface is described by

$$\left(x^1\right)^2+\left(x^2\right)^2-\left(x^3\right)^2=-S^2 \tag{1}$$

where S is independent of the coordinates $x^i\left(i=1,2,3\right)$, but may depend on time [24] and the metric.

$$d\sigma^2=\left(dx^1\right)^2+\left(dx^2\right)^2-\left(dx^3\right)^2 \tag{2}$$

The Cartesian coordinates may be retrieved from the spherical coordinates (radius of the Universe S, inclination ψ, azimuth θ) and clearly Equation 1 admits of the transformation;

$$x^1=S\sinh\psi\cos\theta$$
$$x^2=S\sinh\psi\sin\theta$$
$$x^3=S\cosh\psi$$

From which we find

$$d\sigma^2=S^2[d\psi^2+\sinh^2\psi d\theta^2]$$

Set

$$r=\sinh\psi$$
$$dr=\cosh\psi d\psi$$

Then

$$d\psi^2=\frac{dr^2}{\cosh^2\psi}=\frac{dr^2}{1+\sinh^2\psi}=\frac{dr^2}{1+r^2}$$
$$d\sigma^2=S^2\left[\frac{dr^2}{1+r^2}+r^2d\theta^2\right]$$

The line element for the (2 + 1) spacetime is therefore given by;

$$\begin{aligned} ds^2 &= c^2dt^2 - d\sigma^2 \\ &= c^2dt^2 - S^2\left[\frac{dr^2}{1+r^2} + r^2d\theta^2\right] \end{aligned} \tag{3}$$

For a space of constant positive curvature, the Robertson – Walker metric is derived by considering a 2 – dimensional coordinate space as a space embedded in the 3 – dimensional hypersurface described by the equation;

$$\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 = S^2 \tag{4}$$

where S is a constant and the metric

$$d\sigma^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \tag{5}$$

Setting

$$\begin{aligned} x^1 &= S\sin\psi\cos\theta \\ x^2 &= S\sin\psi\sin\theta \\ x^2 &= S\cos\psi \end{aligned}$$

We have

$$d\sigma^2 = S^2\left[d\psi^2 + \sin^2\psi d\theta^2\right]$$

If we further substitute;

$$\begin{aligned} r &= \sin\psi \\ dr &= \cos\psi d\psi \\ d\psi^2 &= \frac{dr^2}{\cos^2\psi} = \frac{dr^2}{1-\sin^2\psi} = \frac{dr^2}{1-r^2} \text{ , hence} \\ d\sigma^2 &= S^2\left[\frac{dr^2}{1-r^2} + r^2d\theta^2\right] \end{aligned}$$

Where $r = \sin\psi$

The line element of (2 + 1) – dimensional spacetime is therefore given by;

$$\begin{aligned} ds^2 &= c^2dt^2 - d\sigma^2 \\ &= c^2dt^2 - S^2\left[\frac{dr^2}{1-r^2} + r^2d\theta^2\right] \end{aligned} \tag{6}$$

Equations (3) and (6) can be combined into a single expression by introducing a parameter k that takes values ± 1:

$$ds^2 = c^2dt^2 - S^2\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2\right] \tag{7}$$

In analogy with the Robertson – Walker metric, the above metric can be called the Robertson – Walker metric in (2 + 1) – dimensional space time. The metric also describes a 2 – dimensional coordinate space of zero curvature, (that is, a flat 2 – dimensional space), which is the case for k = 0. Thus, when k = 0, we have a space of zero curvature, that is flat space, k = 1 is a space of positive curvature, which is a closed space and k = -1 describes a space of negative curvature, which is an open space.

To determine the curvature of each surface, we use spatial part of the Robertson – Walker metric. The various surfaces and their curvatures are discussed below.

For positive curvature (k = 1), the metric of the spatial part is given by;

$$d\sigma^2 = S^2\left[\frac{dr^2}{1-r^2} + r^2d\theta^2\right] \tag{8}$$

Then $g_{ij} = \begin{pmatrix} \frac{S^2}{1-r^2} & 0 \\ 0 & S^2r^2 \end{pmatrix}$,

Where $(i,j) = (r,\theta) = (1,2)$

Computation of the non – vanishing Christoffel symbols yields;

$$\Gamma^i_{ii} = \frac{1}{2g_{ii}}\cdot\frac{\partial g_{ii}}{\partial x^i}, \qquad \Gamma^1_{11} = \frac{r}{1-r^2}$$

$$\Gamma^i_{ij} = \frac{1}{2g_{ii}}\cdot\frac{\partial g_{ii}}{\partial x^j}, \qquad \Gamma^2_{21} = \frac{1}{r}$$

$$\Gamma^i_{jj} = \frac{1}{2g_{ii}}\cdot\frac{\partial g_{jj}}{\partial x^i}, \qquad \Gamma^1_{22} = -r(1-r^2)$$

For a 2 – D space, the only non – zero components of the curvature tensor

$$R_{jlhk} = \frac{1}{2}\left(\frac{\partial^2 g_{jh}}{\partial x^k\partial x^l} - \frac{\partial^2 g_{lh}}{\partial x^k\partial x^j} + \frac{\partial^2 g_{lk}}{\partial x^h\partial x^j} - \frac{\partial^2 g_{jk}}{\partial x^h\partial x^l}\right) + g_{rs}\left[\Gamma^r_{jh}\Gamma^s_{lk} - \Gamma^r_{jk}\Gamma^s_{lh}\right] \tag{9}$$

The components R_{1212} and R_{2121} with $R_{1212} = R_{2121}$.

Using the Christoffel symbols obtained above and equation 9, we had;

$$R_{1212} = \frac{1}{2}\left(\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 g_{21}}{\partial x^2 \partial x^1} + \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} \right) + g_{11}\left[\Gamma^1_{11}\Gamma^1_{22} - \Gamma^1_{12}\Gamma^1_{21} \right] + g_{22}\left[\Gamma^2_{11}\Gamma^2_{22} - \Gamma^2_{12}\Gamma^2_{21} \right]$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} \right) + g_{11}\left(\Gamma^1_{11}\Gamma^1_{22} \right) - g_{22}\left(\Gamma^2_{12}\Gamma^2_{21} \right)$$

$$\therefore \quad R_{1212} = -\frac{S^2 r^2}{1-r^2} = R_{2121}$$

Now the curvature of a 2 – D space described by the metric g_{ij} is given by the Gaussian curvature

$$K = -\frac{R_{1212}}{g} \tag{10}$$

Where g is the determinant of the metric.

In this case

$$g = \frac{S^4 r^2}{1-r^2} \quad \text{and} \quad K = \frac{S^2 r^2}{1-r^2} \cdot \frac{1-r^2}{S^4 r^2} = \frac{1}{S^2}$$

This proves that the space indeed has a positive curvature.

For a 2 – D space with k = -1, we have

$$d\sigma^2 = S^2\left[\frac{dr^2}{1+r^2} + r^2 d\theta^2 \right] \tag{11}$$

$$g_{ij} = \begin{pmatrix} \frac{S^2}{1+r^2} & 0 \\ 0 & S^2 r^2 \end{pmatrix}$$

If $(i, j) = (r, \theta) = (1, 2)$, the non – vanishing Christoffel symbols are;

$$\Gamma^i_{ii} = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{ii}}{\partial x^i}, \qquad \Gamma^1_{11} = -\frac{r}{1+r^2}$$

$$\Gamma^i_{ij} = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{ii}}{\partial x^j}, \qquad \Gamma^2_{21} = \frac{1}{r}$$

$$\Gamma^i_{jj} = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{jj}}{\partial x^i}, \qquad \Gamma^1_{22} = -r(1+r^2)$$

The only non – zero independent component of the Riemann curvature tensor is given by;

$$R_{1212} = \frac{1}{2}\left(\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 g_{21}}{\partial x^2 \partial x^1} + \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} \right) + g_{11}\left[\Gamma^1_{11}\Gamma^1_{22} - \Gamma^1_{12}\Gamma^1_{21} \right] + g_{22}\left[\Gamma^2_{11}\Gamma^2_{22} - \Gamma^2_{12}\Gamma^2_{21} \right]$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} \right) + g_{11}\left(\Gamma^1_{11}\Gamma^1_{22} \right) - g_{22}\left(\Gamma^2_{12}\Gamma^2_{21} \right)$$

Thus $R_{1212} = \frac{S^2 r^2}{1+r^2}$

This yields for the Gaussian curvature;

$K = -\frac{R_{1212}}{g}$, where $g = \frac{S^4 r^2}{1+r^2}$ is the determinant of the metric.

Hence the curvature of the space is $K = -\frac{S^2 r^2}{1+r^2} \cdot \frac{1+r^2}{S^4 r^2} = -\frac{1}{S^2}$.

Hence a 2 – D space with k = -1 is indeed a space with negative curvature.

Finally, for a 2 – D space with k = 0, that one with line element;

$$d\sigma^2 = S^2\left[dr^2 + r^2 d\theta^2 \right] \tag{12}$$

With metric

$$g_{ij} = \begin{pmatrix} S^2 & 0 \\ 0 & S^2 r^2 \end{pmatrix} \tag{13}$$

The non – vanishing Christoffel symbols are

$$\Gamma^i_{ij} = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{ii}}{\partial x^j}, \qquad \Gamma^2_{21} = \frac{1}{r}$$

$$\Gamma^i_{jj} = \frac{1}{2g_{ii}} \cdot \frac{\partial g_{jj}}{\partial x^i}, \qquad \Gamma^1_{22} = -r$$

The only non – zero independent component.

$$R_{1212} = \frac{1}{2}\left(\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2} - \frac{\partial^2 g_{21}}{\partial x^2 \partial x^1} + \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} \right) + g_{11}\left[\Gamma^1_{11}\Gamma^1_{22} - \Gamma^1_{12}\Gamma^1_{21} \right] + g_{22}\left[\Gamma^2_{11}\Gamma^2_{22} - \Gamma^2_{12}\Gamma^2_{21} \right]$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} \right) - g_{22}\left(\Gamma^2_{12}\Gamma^2_{21} \right)$$

$$= \frac{1}{2}\left(2S^2 \right) - S^2 = 0$$

The Gaussian curvature is given by $K = -\frac{R_{1212}}{g}$ where $g = S^4 r^2$ is the determinant of the metric (13). The Gaussian curvature; K = 0, hence the 2 – D space with k = 0 is indeed a flat space.

3. Results and Discussion

In (3 + 1) – dimensions, the Robertson – Walker metric is given by;

$$ds^2 = c^2 dt^2 - S^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right] \tag{14}$$

From the illustrations above, we observed that in (2 + 1) – dimensions, the Robertson – Walker metric is given by;

$$ds^2 = c^2 dt^2 - S^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 \right]$$

The curvature in both (3 + 1) and (2 + 1) dimensions is determine by the value of the curvature parameter, k. When k = 0, we have a space of zero curvature which is also known as a Euclidean / flat space. The Gaussian curvature; K = 0, also confirms that the space is indeed flat or without any curvature. When k = 1, we have a space of positive curvature which is also known as a closed space. The Gaussian curvature, $K = \frac{1}{S^2}$, proves that the space has a positive curvature. When k = -1, we have a space of negative curvature which is also known as an open space. The Gaussian curvature, $K = -\frac{1}{S^2}$, also proves that the space indeed has a negative curvature. One interesting feature about the (2 + 1) – dimensional universe is that, any 2 – dimensional creature walking on this surface will always see its surface to be flat irrespective of the curvature of the space.

4. Conclusion

In conclusion, the Robertson – Walker metric in (2 + 1) – dimensional space has been derived and can be used as a toy model to study Robertson – Walker metric in (3 + 1) – dimensional space.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Vincenzo De Risi. Mathematizing space: The objects of Geometry from Antiquity to the Early Modern Age. Birkhauser; 2015.

[ISBN 978-3-319-12102-4]

[2] Boyer. Ionia and the pythagoreans. 1991;43.

[3] Tabak John. Geometry: The language of space and form. Infobase Publishing. 2014;xiv.
[ISBN 978-0816049530]

[4] Walter A. Meyer. Geometry and its application. Elsevier; 2006.
[ISBN 978-0-08-047803-6]

[5] Andrew L. An introduction to modern cosmology. Second edition. England: John Wiley & Sons Ltd, Atrium, Southern Gate, Chichester, West Sussex P0198sQ; 2003.

[6] Andrew J. S. Hamilton. General Relativity, Black Hole and Cosmology. 2018;226.
Available:http://jila.colorado.edu/~ajsh/astr3740_17/notes.html

[7] George F. R. Ellis, Roy Maartens, Malcom A. H. MacCallum. Relativistic Cosmology. New York: Cambridge University Press. 2012;201.

[8] Katanaev MO. Cosmological models with homogeneous and isotropic spatial sections. Theoretical and Mathematical Physics. 2017;191(2):661–688.

[9] Mukhanov VF. Physical foundations to cosmology. Cambridge: Cambridge University Press; 2005.

[10] Peter P, Uzan JP. Primordial cosmology. Oxford: Oxford University Press; 2009.

[11] Cosimo Bambi, Alexandre D. Dolgov. Introduction to particle cosmology: The standard model of cosmology and its open problems. Springer Heildelberg New York Dordrecht London. 2016;3.
[ISBN: 978-3-662-48078-6]

[12] Haradhan Kumar Mohajan. Friedmann robertson – Walker (FRW) models in cosmology. Journal of Environmental Treatment Techniques. 2013;158.
[ISSN: 2309-1185]

[13] Barrow JD, Matzner RA. The homogeneity and isotropy of the universe. Monthly Notices of the Royal Astronomical Society. 1977;181:719–729.

[14] Ian Ridpath. A dictionary of astronomy (2 rev. ed.). Current Online Version; 2016.

[15] Blau SK, Guth AH. Inflationary cosmology. In Three Hundred Years of Gravitation, edited by S. W. Hawking and W. Israel, Cambridge University Press. 1987;524–603.

[16] Weinberg S. Gravitation and cosmlogy: Principles and applications of the general theory of relativity. New York: John Wiley and Sons; 1972.

[17] Trodden M, Carroll SM. TASI lectures: Introduction to cosmology. Lectures at the Theoretical Advanced study Institute in elementary particle physics (TASI – 2003, Recent Trends in String Theory, University of Colorado at Boulder, 2003. Edited by J. M. Maldacena; Hackensack, World Scientific. 2005;548.
Available:astro-ph/0401547

[18] Rubakov VA. Introduction to cosmology. (RTN Winter School of Strings, Supergravity and Gauge Theories, SISSA, Trieste, Italy). 2005;58.
Available:pos.sissa.it.

[19] Brandenberger RH. Inflationary cosmology: Progress and problems. Lectures at the International School on Cosmology (Kish Island, Iran. 1999; proceedings: Klumer, Dordrecht). 1999;48. Available:hep-ph/9910410

[20] Matts R. Introduction to cosmology. second edition. England. John Wiley & Sons Ltd, Baffins Lane, Chichester, West Sussex P019 1UD; 1994.

[21] Erica Flapan. Knots, molecules and the universe: An introduction to topology. United States of America. American Mathematical Society. 2016;5.

[22] Richard M. West. Understanding the universe: The impact of space astronomy. Holland. D. Reidel Publishing Company. 1983;161.

[23] A Journey into the Cosmos – FLRW metric and the Friedmann Equation. Physics Forums Insights; 2018.

[24] Jayant Narlikar V. Introduction to cosmology, second edition. England. Cambridge University Press; 1993.

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