Asian Research Journal of Mathematics

16(7): 1-7, 2020; Article no.ARJOM.57020

ISSN: 2456-477X

Existence of Nonoscillation Solutions of Second-order Neutral Differential Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2020/v16i730198

Editor(s):

(1) Dr. Sheng Zhang, Bohai University, China.

Reviewers:

(1) Jackson Akpojaro, University of Africa, Toru-Orua, Nigeria.

(2) Francisco Bulnes, Tecnológico de Estudios Superiores de Chalco, Mexico.

Complete Peer review History: http://www.sdiarticle4.com/review-history/57020

Received:* *10 March 2020

Accepted:* *13 May 2020

Published:* *27 May 2020

Short Research Article

Abstract

This paper is concerned with existence of nonoscillation solution for a family of second-order neutral differential equations with positive and negative coefficients. A sufficient conditions for existence of nonoscillation solution is obtained by contraction fixed point theorem, special case of the equation has also been studied.

Keywords: Differential equation; nonoscillation solutions; existence.

2010 Mathematics Subject Classification: 34K10, 34K11.

1 Introduction

In this paper, we consider existence of nonoscillation solution of second-order neutral differential equations with positive and negative coefficients.

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$$\left(r(t)\left(x(t)+cx(t-\tau)\right)'\right)'+[P(t)x(t-\sigma)-Q(t)x(t-\delta)]=0, \quad t\geqslant t_0 \tag{1.1}$$

where $\tau, \sigma, \delta \in R^+$, $c \in R, c \neq \pm 1$,and $r(t), P(t), Q(t) \in C([t_0, \infty), R^+)$, $R^+ = [0, +\infty)$. Let $\mu = \{\tau, \sigma, \delta\}$. By a solution of equation (1.1), we mean a continuously function $x(t) \in C([t_0 - \mu, \infty), R)$ for some $t_1 \geqslant t_0$, such that $r(t)\left(x(t) + cx(t-\tau)\right)'$ is continuously differentiable on $[t_1, \infty)$ and such that equation (1.1) is satisfied for $t \geqslant t_1$.

Recently, More and more people are interested in nonoscillatory criteria of differential equations. we refer the reader to $[1-11]$, the differetial equation in [1].

$$\frac{d^n}{dt^n}[x(t)+cx(t-\tau)]+(-1)^{n+1}[P(t)x(t-\sigma)-Q(t)x(t-\delta)]=0, \quad t\geqslant t_0$$

studied nonoscillation solution for a family of higher-order neutral differential equations with positive and negative coefficients, Our principal goal in this paper is to derive existence of nonoscillation solutions for equation (1.1).

2 Existence Theorems

Theorem 1. Assume that $0 \leqslant c < 1$ and

$$\int_{t_0}^{\infty}\frac{1}{r(u)}\int_u^{\infty}P(s)dsdu<\infty, \quad \int_{t_0}^{\infty}\frac{1}{r(u)}\int_u^{\infty}Q(s)dsdu<\infty. \tag{2.1}$$

Further, assume that there exists a constant $\alpha > \frac{1}{1-c}$ and a sufficiently large $t_1 \geqslant t_0$ such that

$$Q(t)\geqslant \alpha P(t), \quad for \ \ t\geqslant t_1 \tag{2.2}$$

Then (1.1) has a bounded nonoscillatory solution.

Proof. By (2.1) and (2.2), there exists a t_1 sufficiently large such that

$$c+\int_t^{\infty}\frac{1}{r(u)}\int_u^{\infty}(P(s)+Q(s))dsdu\leqslant\theta_1<1, \quad for \ \ t\geqslant t_1 \tag{2.3}$$

where θ_1 is a constant, and

$$0\leqslant\int_t^{\infty}\frac{1}{r(u)}\int_u^{\infty}(\alpha MQ(s)-MP(s))dsdu\leqslant c-1+\alpha M, \ \ for \ \ t\geqslant t_1 \tag{2.4}$$

hold, where M is positive constant such that

$$\frac{1-c}{\alpha}<M\leqslant\frac{1-c}{1+c\alpha} \tag{2.5}$$

holds. Let X be the set of all continuous and bounded functions on $[t_0, \infty)$ with the norm $\| x \| = sup_{t\geqslant t_0}|x(t)|$, we define a closed bounded subset Ω of X as follows:

$$\Omega=\{x\in X: M\leqslant x(t)\leqslant\alpha M, t\geqslant t_0\}$$

Define an operator $S : \Omega \to X$ as follows:

$$Sx(t)=\begin{cases}1-c-cx(t-\tau)+\int_t^{\infty}\frac{1}{r(u)}\int_u^{\infty}(Q(s)x(s-\delta)-P(s)x(s-\sigma))dsdu, & t\geqslant t_1,\\ Sx(t_1), & t_0\leqslant t\leqslant t_1.\end{cases}$$

We shall show that $S\Omega \subset \Omega$. In fact, for every $x \in \Omega$, and $t \geqslant t_1$, using (2.4) and (2.5) we get

$$\begin{aligned} Sx(t) =&1 - c - cx(t-\tau) + \int_t^\infty \frac{1}{r(u)} \int_u^\infty (Q(s)x(s-\delta) - P(s)x(s-\sigma))dsdu \\ \leqslant&1 - c + \int_t^\infty \frac{1}{r(u)} \int_u^\infty (\alpha M Q(s) - MP(s))dsdu \\ \leqslant&\alpha M \end{aligned}$$

Furthermore, in view of (2.2) and (2.5) we have

$$\begin{aligned} Sx(t) =&1 - c - cx(t-\tau) + \int_t^\infty \frac{1}{r(u)} \int_u^\infty (Q(s)x(s-\delta) - P(s)x(s-\sigma))dsdu \\ \geqslant&1 - c - c\alpha M + \int_t^\infty \frac{1}{r(u)} \int_u^\infty (MQ(s) - \alpha MP(s))dsdu \\ \geqslant&1 - c - c\alpha M \\ \geqslant&M \end{aligned}$$

Thus, we proved that $S\Omega \subset \Omega$.

Now we shall show that operator S is a contraction operator on Ω.

In fact, for $x, y \in \Omega$ and $t > t_1$, we have

$$\begin{aligned} |Sx(t) - Sy(t)| \leqslant& c|x(t-\tau) - y(t-\tau)| + \int_t^\infty \frac{1}{r(u)} \int_u^\infty P(s)|x(s-\sigma) - y(s-\sigma)|dsdu \\ &+ \int_t^\infty \frac{1}{r(u)} \int_u^\infty Q(s)|x(s-\delta) - y(s-\delta)|dsdu \\ \leqslant&[c + \int_t^\infty \frac{1}{r(u)} \int_u^\infty (P(s) + Q(s)]dsdu] \parallel x - y \parallel \\ \leqslant&\theta_1 \parallel x - y \parallel \end{aligned}$$

This implies that

$$\parallel Sx - Sy \parallel \leqslant \theta_1 \parallel x - y \parallel$$

where in view of (2.3), $\theta_1 < 1$,which proves that S is a contraction operator on Ω . Therefore S has a unique fixed point x in Ω, which is obviously a bounded positive solution of equation (1.1). This copletes the proof of Theorem 1.

Theorem 2. Assume that $1 < c < +\infty$ and that (2.1) holds. Further, assume that there exists a constant $\gamma > \frac{c}{c-1}$ and a sufficiently large $t_1 \geqslant t_0$ such that

$$Q(t) \geqslant \gamma P(t), \quad for \ \ t \geqslant t_1 \tag{2.6}$$

Then (1.1) has a bounded nonoscillatory solution.

Proof. By (2.1) and (2.6), there exists a t_1, sufficiently large such that

$$\frac{1}{c}[1 + \int_{t+\tau}^\infty \frac{1}{r(u)} \int_u^\infty (p(s) + Q(s))dsdu] \leqslant \theta_2 < 1, \ \ for \ \ t \geqslant t_1 \tag{2.7}$$

where θ_2 is a constant, and

$$0 \leqslant \frac{1}{c} \int_{t+\tau}^\infty \frac{1}{r(u)} \int_u^\infty (\gamma M_1 Q(s) - M_1 P(s))dsdu \leqslant 1 - c + c\gamma M_1, \quad for \ \ t \geqslant t_1 \tag{2.8}$$

hold, where M_1 is positive constant such that

$$\frac{c-1}{\gamma c} < M_1 < \frac{c-1}{\gamma + c} \tag{2.9}$$

holds. Let X be the set of all continuous and bounded functions on $[t_0, \infty)$ with the norm $\parallel x \parallel = sup_{t \geqslant t_0}|x(t)|$, we define a closed bounded subset Ω of X as follows

$$\Omega = \{x \in X : M_1 \leqslant x(t) \leqslant \gamma M_1, t \geqslant t_0\}$$

Define an operator $S : \Omega \to X$ as follows

$$Sx(t) = \begin{cases} 1 - \frac{1}{c} - \frac{1}{c}x(t+\tau) + \frac{1}{c}\int_{t+\tau}^{\infty} \frac{1}{r(u)} \int_u^{\infty}(Q(s)x(s-\delta) - P(s)x(s-\sigma))dsdu, & t \geqslant t_1, \\ Sx(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

We shall show that $S\Omega \subset \Omega$. In fact,for every $x \in \Omega$, and $t \geqslant t_1$, using (2.8) and (2.9) we get

$$\begin{aligned} Sx(t) =& 1 - \frac{1}{c} - \frac{1}{c}x(t+\tau) + \frac{1}{c}\int_{t+\tau}^{\infty} \frac{1}{r(u)} \int_u^{\infty}(Q(s)x(s-\delta) - P(s)x(s-\sigma))dsdu \\ \leqslant& 1 - \frac{1}{c} + \frac{1}{c}\int_{t+\tau}^{\infty} \frac{1}{r(u)} \int_u^{\infty}(\gamma M_1 Q(s) - M_1 P(s))dsdu \\ \leqslant& \gamma M_1 \end{aligned}$$

Furthermore, in view of (2.6) and (2.9) we have

$$\begin{aligned} Sx(t) =& 1 - \frac{1}{c} - \frac{1}{c}x(t+\tau) + \frac{1}{c}\int_{t+\tau}^{\infty} \frac{1}{r(u)} \int_u^{\infty}(Q(s)x(s-\delta) - P(s)x(s-\sigma))dsdu \\ \geqslant& 1 - \frac{1}{c} - \frac{\gamma M_1}{c} + \frac{1}{c}\int_{t+\tau}^{\infty} \frac{1}{r(u)} \int_u^{\infty}(M_1 Q(s)) - \gamma M_1 P(s))dsdu \\ \geqslant& 1 - \frac{1}{c} - \frac{\gamma M_1}{c} \\ \geqslant& M_1 \end{aligned}$$

Thus, we proved that $S\Omega \subset \Omega$. Now we shall show that operator S is a contraction operator on Ω. In fact, for $x, y \in \Omega$ and $t > t_1$, we have

$$\begin{aligned} |Sx(t) - Sy(t)| \leqslant& \frac{1}{c}|x(t+\tau) - y(t+\tau)| + \frac{1}{c}\int_{t+\tau}^{\infty} \frac{1}{r(u)} \int_u^{\infty} p(s)|x(s-\sigma) - y(s-\sigma)|dsdu \\ &+ \frac{1}{c}\int_{t+\tau}^{\infty} \frac{1}{r(u)} \int_u^{\infty} Q(s)|x(s-\delta) - y(s-\delta)|dsdu \\ \leqslant& \frac{1}{c}[1 + \int_{t+\tau}^{\infty} \frac{1}{r(u)} \int_u^{\infty}(p(s) + Q(s))dsdu] \parallel x - y \parallel \\ \leqslant& \theta_2 \parallel x - y \parallel \end{aligned}$$

This implies that

$$\parallel Sx - Sy \parallel \leqslant \theta_2 \parallel x - y \parallel$$

where in view of (2.7), $\theta_2 < 1$, which proves that S is a contraction operator on Ω. Therefore S has a unique fixed point x in Ω, which is obviously a bounded positive solution of equation (1.1). This copletes the proof of Theorem 2.

Theorem 3. Assume that $-1 < c < 0$ and that (2.1) holds. Further, assume that there exists a constant $\beta > 1$ and a sufficiently large $t_1 \geqslant t_0$ such that

$$Q(t) \geqslant \beta P(t), \quad for \ \ t \geqslant t_1 \tag{2.10}$$

Then (1.1) has a bounded nonoscillatory solution.

Proof. By (2.1) and (2.10), there exists a t_1 sufficiently large such that

$$-c+\int_t^\infty \frac{1}{r(u)}\int_u^\infty (p(s)+Q(s))dsdu \leqslant \theta_3 < 1, \ \ for \ \ t \geqslant t_1 \tag{2.11}$$

where θ_3 is a constant, and

$$0 \leqslant \int_t^\infty \frac{1}{r(u)}\int_u^\infty (\beta M_2 Q(s) - M_2 P(s))dsdu \leqslant (c+1)(\beta M_2 - 1), \quad for \ \ t \geqslant t_1 \tag{2.12}$$

hold, where M_2 is positive constant such that

$$\frac{1}{\beta} < M_2 \leqslant 1 \tag{2.13}$$

holds. Let X be the set of all continuous and bounded functions on $[t_0, \infty)$ with the norm $\| x \| = sup_{t \geqslant t_0}|x(t)|$, we define a closed bounded subset Ω of X as follows

$$\Omega = \{x \in X : M_2 \leqslant x(t) \leqslant \beta M_2, t \geqslant t_0\}$$

Define an operator $S : \Omega \to X$ as follows

$$Sx(t) = \begin{cases} 1 + c - cx(t-\tau) + \int_t^\infty \frac{1}{r(u)}\int_u^\infty (Q(s)x(s-\delta) - P(s)x(s-\sigma))dsdu, & t \geqslant t_1, \\ Sx(t_1), & t_0 \leqslant t \leqslant t_1. \end{cases}$$

We shall show that $S\Omega \subset \Omega$. In fact, for every $x \in \Omega$, and $t \geqslant t_1$, using (2.12) and (2.13) we get

$$\begin{aligned} Sx(t) =& 1 + c - cx(t-\tau) + \int_t^\infty \frac{1}{r(u)}\int_u^\infty (Q(s)x(s-\delta) - P(s)x(s-\sigma))dsdu \\ \leqslant& 1 + c - c\beta M_2 + \int_t^\infty \frac{1}{r(u)}\int_u^\infty (\beta M_2 Q(s) - M_2 P(s))dsdu \\ \leqslant& 1 + c - c\beta M_2 + (c+1)(\beta M_2 - 1) \\ =& \beta M_2 \end{aligned}$$

Furthermore, in view of (2.10) and (2.13) we have

$$\begin{aligned} Sx(t) =& 1 + c - cx(t-\tau) + \int_t^\infty \frac{1}{r(u)}\int_u^\infty (Q(s)x(s-\delta) - P(s)x(s-\sigma))dsdu \\ \geqslant& 1 + c - cM_2 + \int_t^\infty \frac{1}{r(u)}\int_u^\infty (M_2 Q(s)) - \beta M_2 P(s))dsdu \\ \geqslant& 1 + c - cM_2 \\ \geqslant& M_2 \end{aligned}$$

Thus, we proved that $S\Omega \subset \Omega$. Now we shall show that operator S is a contraction operator on Ω. In fact, for $x, y \in \Omega$ and $t > t_1$, we have

$$\begin{aligned}|Sx(t)-Sy(t)| \leqslant & -c|x(t+\tau)-y(t+\tau)| + \int_t^\infty \frac{1}{r(u)} \int_u^\infty p(s)|x(s-\sigma)-y(s-\sigma)|dsdu \\ & + \int_t^\infty \frac{1}{r(u)} \int_u^\infty Q(s)|x(s-\delta)-y(s-\delta)|dsdu \\ \leqslant & [-c + \int_t^\infty \frac{1}{r(u)} \int_u^\infty (p(s)+Q(s))dsdu] \parallel x-y \parallel \\ \leqslant & \theta_3 \parallel x-y \parallel\end{aligned}$$

This implies that

$$\parallel Sx - Sy \parallel \leqslant \theta_3 \parallel x-y \parallel$$

where in view of (2.11), $\theta_3 < 1$, which proves that S is a contraction operator on Ω. Therefore S has a unique fixed point x in Ω, which is obviously a bounded positive solution of equation (1.1). This copletes the proof of Theorem 3.

Theorem 4. Assume that $-\infty < c < -1$ and that (2.1) holds. Further, assume that there exists a constant $h > 1$ and a sufficiently large $t_1 \geqslant t_0$ such that

$$Q(t) \geqslant hP(t), \quad for \;\; t \geqslant t_1 \tag{2.14}$$

Then (1.1) has a bounded nonoscillatory solution.

Proof : The proof is similar to Theorem 2, we omitted.

By Theorems 1-4, we have the following result

Corollary 1. Assume that $c \in R, c \neq \pm 1$ and

$$\int_{t_0}^\infty \frac{1}{r(u)} \int_u^\infty Q(s)dsdu < \infty.$$

then the neutral differential equation

$$\left(r(t)\left(x(t)+cx(t-\tau)\right)'\right)' - Q(t)x(t-\delta) = 0, \quad t \geqslant t_0$$

has a bounded nonoscillatory solution.

3 Conclusion

In this paper we have introuduced existence of nonoscillatory solutions of differential equations of (1.1), the obtained results are easily applicable, we can find nonoscillatory solutions of higher-order neutral differential equations by contraction fixed point theorem in the future work.

Acknowledgements

This work was supported social science planning support project of Qinghai Province (Nos. 16021).

Competing Interests

Authors have declared that no competing interests exist.

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