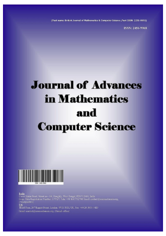

Journal of Advances in Mathematics and Computer Science

35(9): 73-89, 2020; Article no.JAMCS.64666

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

On the Identification of Coefficient and Source Parameters in Elliptic Systems Modelled with Many Boundary Values Problems

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/JAMCS/2020/v35i930324

Editor(s):

(1) Dr. Andrej V. Plotnikov, Odessa State Academy of Civil Engineering and Architecture, Ukraine.

Reviewers:

(1) Ahmad Hussain, King Abdulaziz University, KSA.

(2) Abdolrahman Razani, Imam Khomeini International University, Iran.

Complete Peer review History: http://www.sdiarticle4.com/review-history/64666

Received: 24 November 2020

Accepted: 29 December 2020

Published: 30 December 2020

Original Research Article

Abstract

The inverse problem for determination of parameters related to the support and/or functions describing the intensity of coefficient and sources in models based strongly elliptic second order systems is posed with Cauchy data over specification at boundary. This stablish a set of various boundary value problems associated with the same group of unknown parameters. A Lipschitz boundary dissection is used for decomposing each Cauchy data into pairs of complementary mixed boundary values problems. The concept of Calderon projector is introduced as a tool to check the consistency of the Cauchy data and to demonstrate the equivalence of these two problems. This lets you define a discrepancy function to measure the distance between the solutions of problems obtained by dissecting Lipschitz Cauchy data. This discrepancy appears as a consequence of inadequate parameters values in the constitutive relations. For Cauchy noisy data, the difference between these solutions would be small if the parameters used in the solution are correct. The methodology we propose explores concepts as Lipschitz Boundary Dissection, Complementary Mixed Problems with trial parameters and Internal Discrepancy fields. Differentiable and non-differentiable optimizations algorithms can then be used in the reconstruction of these parameters simultaneously. Numerical experiments are presented.

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Keywords: *Inverse problems; coefficients and source reconstruction; elliptic systems; Lipschitz Dissection; Calderon projector; internal discrepancy.*

1 Introduction

It is possible to find systems of linear elliptic partial differential equations in most of the engineering applications. Stationary multiphysics models related with applications in structural mechanics, heat transfer, convection and diffusion, electromagnetic, fluid dynamics and acoustic are the main examples of represented by the following system, in which the unknown fields may be a vector To find solution $u(x)$ and the multiplier $\lambda(x)$ such that

$$\begin{cases} \nabla.(-c\nabla u - \alpha u + \gamma) + \beta.\nabla u + au = f & \text{if } x \in \Omega; \\ hu = g & \text{if } x \in \partial\Omega_D \cup \partial\Omega_N; \\ \nu.(c\nabla u + \alpha u - \gamma) + qu = g_\nu - h^*\lambda & \text{if } x \in \partial\Omega_N \cup \partial\Omega_D; \end{cases} \tag{1.1}$$

where ν is the outward unit normal vector on $\partial\Omega := \partial\Omega_D \cup \Pi \cup \partial\Omega_N$. The presence of multiplier generalizes boundaries conditions and adjust the solution for the case of overprescripion of boundary data. The coefficients $(c, \alpha, \gamma, \beta, a, h)$ can be represented by matrices and vectors accordingly, and are modelled as given information about the physics of the model. In the context of the continuous thermodynamics field theory, these functions are associated with the constitutive relations that resolves material ambiguity of balance equations. If parameters in these functions are correct, then experientially measured values of field u and extensive or intensives quantities related with it will be correctly determinate by solving, analytically or numerically, the system (1.1) and posprocessing the solution. Meanwhile, if these constitutive parameters are incorrect, we can expect a discrepancy between the experimentally measured data and the theoretically calculated one. When there exist overprescripition of Cauchy data to compensate the parameter indetermination, complementary problems with trial parameters also will present an internal discrepancy in the fields values. Before we start the formulation of the inverse parameter problem studied here, let us review some basic concepts.

1.1 Operators representation for the system of equations (1.1)

In order to give a more concise notation to problem (1.1), let us group its coefficients in a set of matrix functions $(A_{jk}, A_j, A) : \Omega \to \mathbf{R}^{m\times m}$ and rewrite the system operator as

$$\mathcal{L}u = -\sum_{j=1}^{d}(\sum_{k=1}^{d}\partial_j(A_{jk}\partial_k)u + A_j\partial_j u) + Au \ . \tag{1.2}$$

Then u is a column vector with m scalar fields and $\mathcal{L}u : \Omega \to \mathbf{R}^m$. The main part of the operator will be

$$\mathcal{L}_0 u = -\sum_{j=1}^{d}\partial_j\mathcal{B}_j u \text{ where } \mathcal{B}_j = \sum_{k=1}^{d}A_{jk}\partial_k \tag{1.3}$$

most of the constitutive theory prepare the system to be strongly ellptic For engineers this this means a good system producing a well posed direct problems. We say that a differential operator $\mathcal{L}$ is strongly elliptic on Ω if

$$Re\sum_{j=1}^{d}\sum_{k=1}^{d}(A_{jk}(x)\xi_k\eta)^*\xi_j\eta \geq c|\xi|^2|\eta|^2 \text{ for all } x \in \Omega\ ,\ \xi \in \mathbf{R}^d \text{ and } \eta \in \mathbf{C}^m\ . \tag{1.4}$$

We complete the problem characterization by supposing that Ω is a Lipschitz domain, which means in $\mathbf{R}^2$ that it is polygonal $C^{0,1}$. By saying that γ is the trace operator and that the trace conormal derivative is

$$\mathcal{B}_\nu u = \sum_{j=1}^{d} \nu_j \gamma[\mathcal{B}_j u] \tag{1.5}$$

More information about the elliptic system can be found in [1].

1.2 Direct problem with strongly elliptic operators

Let Ω a domain with Lipschitz dissection boundary $\partial\Omega = \partial\Omega_N \cup \Pi \cup \partial\Omega_N$. The mixed boundary value problem for the physical model given by (1.1) is given by the well posed problem $P^{\alpha}_{f_\alpha, g_D, g_N}$: To find $u \in H^1(\Omega)^m$ such that

$$P^{\alpha}_{f_\alpha, g_D, g_N} \begin{cases} \mathcal{L}_\alpha u = f_\alpha & \text{if } x \in \Omega; \\ \gamma[u] = g_D & \text{if } x \in \partial\Omega_D; \\ \mathcal{B}^{\alpha}_\nu u = g_N & \text{if } x \in \partial\Omega_N; \end{cases} \tag{1.6}$$

Note that we make the set of parameters α associated with sources and constitutive equations explicit to reinforce the fact that we are interested the parameter determination problem. It is also possible to show that (1.6) has the following weak formulation $W^{\alpha}_{f_\alpha, g_D, g_N}$

$$\begin{cases} (\mathcal{L}_\alpha u, v)_\Omega + (\mathcal{B}^{\alpha}_\nu u, \gamma[v])_{\partial\Omega} = \Phi_\alpha(u, v) = & \\ \quad = (f_\alpha, v)_\Omega + (g_N, \gamma[v])_{\partial\Omega_N} & \text{if } v \in H^1_D(\Omega)^m; \\ \gamma[u] = g_D & \text{if } x \in \partial\Omega_D. \end{cases} \tag{1.7}$$

This weak formulation is used for computational implementation of the Galerkin finite elements method [2] for mixed problems. Sobolev space definition for spaces $H^1(\Omega)$, $H^1_D(\Omega)$ ant its trace operators are conventional and can be found in [1], [2], [3].

2 Boundary Integral Equation Methodology in Inverse Problems

A second set equations associated with problem (1.1) uses the second Green's identity to introduce the volume, single and double layer potentials [4], [5], [3] and the Boundary integral formulation. It depends on the possibility of determination of the fundamental solution, whose existence is well established when coefficients are smooth or even Lipschitz. But in such a way that even in situations where no explicit functional expression for it can be obtained it can be used for analysis of the inverse problem. In this case methods for solution of the direct problem such as finite elements combined with the optimization procedure proposed here will be appropriated for the solution of the inverse parameter problem. One second important scenario occurs when we know the fundamental solution of the elliptic systems explicitly, and consequentially, we also know the fundamental solutions dependence on parameters. In this case the method of fundamental solutions can be used to numerically determine the Green's functions and construct solutions of associated mixed problems with Dirichlet and Neumann data prescribed on the complementary parts of the boundary and to give explicit numerical solutions for the involved complementary problems. The construction of explicit schemes for the parameters determination is then also possible. In this work we will only presents results for the case in which we don't now explicitly the fundamental solution.

2.1 Fundamental solution

Definition 1. *A Fundamental solution to the operator* $\mathcal{L}^*_\alpha$ *is a distributional solution, [1] [6], of the equation:*

$$\mathcal{L}^*_\alpha u^\alpha_\xi = \delta_\xi \in \mathcal{E}^*(\mathbf{R}^d)^m := ((C^\infty(\Omega))^*)^m \tag{2.1}$$

where δ_ξ *is the delta Dirac operator defined as* $\delta_\xi[\phi] = \phi(\xi)$*, for* $\xi \in \Omega$ *and* $\phi \in (\mathcal{D}(\Omega))^m$ *and* $\mathcal{L}^*_\alpha$ *is the adjoint operator.*

The fundamental solution is the singular distribution with pole at $\xi \in \mathbf{R}^d$, $u^\alpha_\xi = G^\alpha_\xi(x)$, satisfying the fundamental equation (2.1). Its depends on constitutive functions parameters but not on the sources parameters. A general solution of this equation is the sum of a regular solution and the fundamental solution

$$u^\alpha_\xi(x) = v_\alpha(x) + G^\alpha_\xi(x) \ , \ x \in \overline{\Omega} \ . \tag{2.2}$$

The regular solution v_α is in the space, $H_{\mathcal{L}^*}(\Omega) = \{v \in L^2(\Omega) \text{ such that } \mathcal{L}^* v = 0\}$. It is an $\mathcal{L}^*_\alpha$-harmonic function which is an homogeneous solution of the $\mathcal{L}^*_\alpha$-operator equation. The following Lemma is important in the formulation of the boundary integral equation for the model based on operator (1.2)

Lemma 1. *Let* Ω *a Lipschitz domain,* u *and* $v \in H^1(\Omega)^m$*, let the coefficients* A^α_{jk}*,* A^α_j *and* A^α *be* $L_\infty(\Omega)^{m\times m}$ *functions,* $(.,.)_\Omega$ *and* $(.,.)_{\partial\Omega}$*, be respectively the duality pairs in* Ω *and in its* $\partial\Omega$ *and* $\mathcal{L}_\alpha u = f_\alpha$*, if additionally:*

(i) A^α_{jk} *are Lipschitz and* $u \in H^2(\Omega)^m$*, then*

$$\Phi_\alpha(u, v) = (\mathcal{L}_\alpha u, v)_\Omega + (\mathcal{B}^\alpha_\nu u, \gamma[v])_{\partial\Omega}; \tag{2.3}$$

(ii) A^α_{jk} *and* A^α_j *are Lipschitz and* $v \in H^2(\Omega)^m$*, then*

$$\Phi_\alpha(u, v) = (u, \mathcal{L}^*_\alpha v)_\Omega + (\gamma[u], \tilde{\mathcal{B}^\alpha_\nu} v)_{\partial\Omega}; \tag{2.4}$$

(iii) $\mathcal{L}_\alpha u \in L_2(\Omega)^m$*, then the first Green identity (2.3) is verified for* u *and* $v \in H^1(\Omega)^m$*;*

(iv) $\mathcal{L}^*_\alpha u \in L_2(\Omega)^m$*, then the first adjoint Green identity (2.4) is verified for* u *and* $v \in H^1(\Omega)^m$*;*

(v) both $\mathcal{L}_\alpha u$ *and* $\mathcal{L}^*_\alpha u \in L_2(\Omega)^m$*, then the second Green identity*

$$(\mathcal{L}_\alpha u, v)_\Omega - (u, \mathcal{L}^*_\alpha v)_\Omega = (\gamma[u], \tilde{\mathcal{B}^\alpha_\nu} v)_{\partial\Omega} - (\mathcal{B}^\alpha_\nu u, \gamma[v])_{\partial\Omega} \tag{2.5}$$

is verified for u *and* $v \in H^1(\Omega)^m$*;*

(vi) $\mathcal{L}_\alpha u = f_\alpha in\Omega$ *and* $f_\alpha \in \tilde{H}^{-1}(\Omega)^m$*, then there exist* $g \in H^{-\frac{1}{2}}(\Omega)^m$ *such that*

$$\Phi_\alpha(u, v) = (f_\alpha, v)_\Omega + (g, v)_{\partial\Omega} \text{ for } v \in H^1(\Omega)^m. \tag{2.6}$$

(vii) Furthermore, g *is uniquely determined by both* u *and* f*, and note by* u *alone, and we have the estimate*

$$\|g\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le C\|u\|_{H^1(\Omega)^m} + C\|f\|_{\tilde{H}^{-1}(\partial\Omega)^m} \tag{2.7}$$

Proof: See Lemmas (4.1), (4.2), (4.3)and (4.4) in [1].

2.2 Boundary integral equation, reciprocity gap equation and green's function equation

If we adapt result (2.5) to use the field $v(x) = G^\alpha_\xi(x)$ it will result the following *boundary integral equation*

$$u(\xi) = (f_\alpha(.), G^\alpha_\xi(.))_\Omega - (\gamma[u](.), \tilde{\mathcal{B}^\alpha_\nu}[G^\alpha_\xi(.)])_{\partial\Omega} + (\mathcal{B}^\alpha_\nu u, \gamma[G^\alpha_\xi(.)])_{\partial\Omega} \tag{2.8}$$

$\xi \in \Omega$. If alternatively the regular $v_\alpha(x) \in \mathcal{L}^*_\alpha$-harmonic is introduced, we obtain the *reciprocity gap equation*

$$0 = (f_\alpha(.), v_\alpha(.))_\Omega - (\gamma[u](.), \tilde{\mathcal{B}^\alpha_\nu}[v_\alpha(.)])_{\partial\Omega} + (\mathcal{B}^\alpha_\nu u, \gamma[v_\alpha(.)])_{\partial\Omega} \tag{2.9}$$

Note that $G^\alpha_\xi(x) \in \Omega$ only is singular if $\xi \in \Omega$. If not, it becomes regular and so, it is also a $\mathcal{L}^*_\alpha$-harmonic. Fields v_α and G^α_ξ can be have the traces at the boundary $\partial\Omega$ adjust to give Green's functions for the mixed strongly elliptic problem (1.6).

$$P^\alpha_{\delta_\xi, g_D, g_N} \begin{cases} \mathcal{L}^*_\alpha[v^\xi_\alpha + G^\alpha_\xi] = \delta_\xi & \text{if } x, \xi \in \Omega; \\ \gamma[v^\xi_\alpha + G^\alpha_\xi] = 0 & \text{if } x \in \partial\Omega_D; \\ \tilde{\mathcal{B}^\alpha_\nu}[v^\xi_\alpha + G^\alpha_\xi] = 0 & \text{if } x \in \partial\Omega_N; \end{cases} \tag{2.10}$$

$$u(\xi) = (f_\alpha(.), v_\alpha(.) + G^\alpha_\xi(.))_\Omega - (\gamma[u](.), \tilde{\mathcal{B}^\alpha_\nu}[v_\alpha(.) + G^\alpha_\xi(.)])_{\partial\Omega} + (\mathcal{B}^\alpha_\nu u, \gamma[v_\alpha(.) + G^\alpha_\xi(.)])_{\partial\Omega} \tag{2.11}$$

is the *Green's boundary integral equation for the mixed problem.*

The Boundary Integral Equation is composed of three potentials. The volumetric or Newtonian

$$\mathcal{N}_\alpha[f_\alpha](\xi) := (f_\alpha(.), G^\alpha_\xi(.))_\Omega = \int_\Omega G^\alpha_\xi(x) f_\alpha(x) dx \text{ for } \xi \in \Omega \ , \tag{2.12}$$

the single layer potential

$$SL[\mathcal{B}_\nu[u]](\xi) := (\mathcal{B}^\alpha_\nu[u], \gamma[G^\alpha_\xi(.)])_{\partial\Omega} = \int_{\partial\Omega} \gamma_x G^\alpha_\xi(x) \mathcal{B}^\alpha_\nu[u] ds_x \text{ for } \xi \in \Omega \tag{2.13}$$

and the double layer potential

$$DL[u](\xi) := ([\gamma u](.), \tilde{\mathcal{B}^\alpha_\nu}[G^\alpha_\xi(.)])_{\partial\Omega} = \int_{\partial\Omega} \tilde{\mathcal{B}^\alpha_\nu} G^\alpha_\xi(x) \gamma_x[u] ds_x \text{ for } \xi \in \Omega \ . \tag{2.14}$$

They are extensions integral operators to the interior of Ω. The first propagates the influence of sources at a given point x into another point ξ, both in Ω, and the second and the third propagates the Dirichlet and in the Neumann parts Cauchy data at the interface boundary $\partial\Omega$ to the interior of Ω.

The Newtonian potential satisfies the inhomogeneous equation and so, it is in the manifold

$$\mathcal{N}_\alpha[f_\alpha]) \in H^1_{\mathcal{L}^*_\alpha, f_\alpha} := \{v \in H^1(\Omega) | \mathcal{L}^*_\alpha v = f_\alpha\} \tag{2.15}$$

The double and the single layer potentials satisfies the homogeneous when ξ is not in $\partial\Omega$, and so, they are $\mathcal{L}^*_\alpha$-harmonic functions, and so

$$SL[g_\nu], DL[g] \in H^1_{\mathcal{L}^*_\alpha, 0} := \{v \in H^1(\Omega) | \mathcal{L}^*_\alpha v = 0\}. \tag{2.16}$$

Since they all are dependent on the parameters on α, for different trial parameters they will do different extensions of Cauchy data.

Remark 1. *In the distributional framework, [6], the system solutions fields, constitutive coefficients fields and sources fields can be extended for the whole space* $\mathbf{R}^d$ *([7],[8]).*

If we denote $\mathbf{R}^d = \Omega^- \cup \partial\Omega \cup \Omega^+$, *where* $\Omega^- := \Omega$ *and* Ω^+ *the exterior of* $\Omega \cup \partial\Omega$, *the derivation of the third Green's identity is straightforward When* $u_\alpha = u^+_\alpha + u^-_\alpha \in L^2(\mathbf{R}^d)^m$, *with* $u^\pm_\alpha \in H^1(\Omega^\pm)^m$ *and* u *has appropriated asymptomatic behaviour, and* $f_\alpha = f^+_\alpha + f^-_\alpha \in H^{-1}(\mathbf{R}^d)^m$, *has compact support, we can enunciate the Third Green Identity [1]*

$$u(\xi) = (f_\alpha(.), G^\alpha_\xi(.))_\Omega - ([\gamma u](.), \tilde{\mathcal{B}^\alpha_\nu}[G^\alpha_\xi(.)])_{\partial\Omega} + ([\mathcal{B}^\alpha_\nu u], \gamma[G^\alpha_\xi(.)])_{\partial\Omega} \tag{2.17}$$

for $\xi \in \Omega$ *and where* $[u]$ *and* $[\mathcal{B}^\alpha_\nu u]$ *are the jump observed in the second order equation Cauchy data at the boundary* $\partial\Omega$. *Since our main interest is in the interior problem, we will not explore those possibilities.*

2.3 Green's function Methodology for extension with mixed data

Let us consider a well posed mixed boundary value problem (2.10) for which we consider that we already have determined the fundamental solution $G_x^\alpha(x)$ and the we solve the homogeneous source mixed problem problem ,

$$P^\alpha_{0,-G_\xi^\alpha(x)|_{\Gamma_D},-\tilde{\mathcal{B}}_\nu[G_\xi^\alpha](x)|_{\Gamma_N}}$$

and obtain a solution $v_{\alpha.\Gamma_D,\Gamma_N}(x)$, which exist and is unique since the operator is strongly elliptic.

By perturbing the fundamental solution, G_ξ^α with this $v_{\alpha.\Gamma_D,\Gamma_N}(x)$ solution of the auxiliary problem we have formally the extension operators for the mixed problem based on the new Green's fundamental solution for the mixed problem.

$$G_\xi^{\alpha,\Gamma_D,\Gamma_N}(x) = G_\xi^\alpha(x) + v_{\alpha,\Gamma_D,\Gamma_N}(x)$$

The Green's inverse for the mixed problem will be

$$u(\xi) = \int_\Omega G_\xi^{\alpha,\Gamma_D,\Gamma_N}(x)f(x)dx - DL^{\alpha,\Gamma_D,\Gamma_N}[g^D](\xi) + SL^{\alpha,\Gamma_D,\Gamma_N}[g_\nu^N](\xi)\ ,\ \xi \in \Omega. \tag{2.18}$$

It is not difficult to see that those three integral operators propagates only part of the Cauchy data. Dirichlet data in Γ_D and Neumann data in Γ_N, and consequentely, it can be viewed as a formal Green's inverse for the mixed problem.

2.4 The Calderon projector and the integral Equations based Methodologies for mixed data problem solution

The Boundary integral equation (2.8) are compatible with the applications of the traces operators at the boundary.

$$\gamma_\xi[u](\xi) = \gamma_\xi \int_\Omega G_\xi^\alpha(x)f_\alpha(x)dx + \gamma_\xi SL^\alpha[\mathcal{B}_\nu^\alpha[u]](\xi) - \gamma_\xi DL^\alpha[u](\xi)$$

$$\mathcal{B}_\xi^\alpha[u](\xi) = \mathcal{B}_\xi^\alpha \int_\Omega G_\xi^\alpha(x)f_\alpha(x)dx + \mathcal{B}_\xi^\alpha SL^\alpha[\mathcal{B}_\nu^\alpha[u]](\xi) - \mathcal{B}_\xi^\alpha DL^\alpha[u](\xi)$$

For zero source this system of integral equations can be arranged as a linear operator, that can be proved to be a projector. It is the Calderon's operator:

Definition 2. *The Calderón operator is the* 2×2 *linear operator* $\mathcal{C}^\alpha : (H^{\frac{1}{2}}(\Omega))^m \times (H^{-\frac{1}{2}}(\Omega))^m \to (H^{\frac{1}{2}}(\Omega))^m \times (H^{-\frac{1}{2}}(\Omega))^m$ *defined by*

$$\mathcal{C}^\alpha[\gamma u, \mathcal{B}_\nu^\alpha]^T = \begin{bmatrix} -\gamma DL^\alpha[\gamma u] & \gamma SL^\alpha[\mathcal{B}_\nu u] \\ -\mathcal{B}_\nu^\alpha DL^\alpha[\gamma u] & \mathcal{B}_\nu^\alpha SL^\alpha[\mathcal{B}_\nu^\alpha u] \end{bmatrix}$$

The integral operators in this system that are singular can have it singular part explicitly separated in order to rewrite the system in terms of index zero Fredholm Boundary Operators, that is

(i) $S^\alpha_{x\to\xi} = \gamma_\xi SL^\alpha_{x\to\xi} : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$;

(ii) $\tilde{T}^{\alpha *}_{x\to\xi} = -I_{x\to\xi} + 2\mathcal{B}^\alpha_{\nu_\xi} SL^\alpha_{x\to\xi} : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$;

(iii) $T^\alpha_{x\to\xi} = I_{x\to\xi} + 2\gamma_\xi DL^\alpha_{x\to\xi} : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$;

(iv) $-R^\alpha_{x\to\xi} = \mathcal{B}^\alpha_{\nu_\xi} DL_{x\to\xi} : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$.

The index zero Fredholm representation of Calderon operator is

$$\mathcal{C}^\alpha = \begin{bmatrix} \frac{1}{2}(I - T^\alpha) & S^\alpha \\ R^\alpha & \frac{1}{2}(I + \tilde{T}^{\alpha *}) \end{bmatrix}.$$

It's not dificult to shown that Calderon's operator is a projector, that is: $\mathcal{C}^\alpha[g, g_\nu] = [g, g_\nu]^T = (\mathcal{C}^\alpha)^2[g, g_\nu]^T$. As a projector, the Calderon's operator imposes restriction on the Cauchy data. The Dirichlet part may be used to determines de Neumamm part and vice-versa. Also Lipschitz dissections of the boundary may be used for Cauchy data partition with the formulation of complementary mixed problems. Since it depends on the constitutive parameters, the Cauchy data must be consistent with the constitutive parameters that generates it. For the case of wrong trial parameters, inconsistency in complementary mixed problems shroud be expected.

When the mixed boundary value problem is posed with a non null source, $P_{f_\alpha, g^D, g_\nu^N}$, we have an additional term due to the volumetric potential to be added to the Calderón operator and the consequent inclusion of sources parameters

$$\begin{bmatrix} \gamma u(\xi) \\ \mathcal{B}_\nu^\alpha u(\xi) \end{bmatrix} = \begin{bmatrix} \int_\Omega \gamma_\xi [G_\xi^\alpha](y) f(y) dy \\ \int_\Omega \mathcal{B}_{\nu_\xi}^\alpha [G_\xi^\alpha](y) f(y) dy \end{bmatrix} +$$

$$\begin{bmatrix} \frac{1}{2}(I_{x\to\xi} - T_{x\to\xi}^\alpha) & S_{x\to\xi}^\alpha \\ R_{x\to\xi}^\alpha & \frac{1}{2}(I_{x\to\xi} + T_{x\to\xi}^\alpha) \end{bmatrix} \begin{bmatrix} \gamma u(\xi) \\ \mathcal{B}_\nu^\alpha u(\xi) \end{bmatrix}, \; (x, \xi) \in \Gamma \times \Gamma.$$

The Boundary integral equation methodology for the mixed boundary value problem , P_{f, g^D, g_ν^N}, explores this consistence dependence of Cauchy data on constitutives and source parameters.

The method is stated by using the knowledge part of Cauchy data $\gamma u|_{\Gamma_D} = g^D$ and $\mathcal{B}_\nu u|_{\Gamma_N} = g_\nu^N$ to determine the part of Cauchy data that we don't know $\gamma u|_{\Gamma_N} = g^N$ and $\mathcal{B}_\nu u|_{\Gamma_D} = g_\nu^D$.

We have to solve an system like this:

$$\begin{bmatrix} S_{\xi\to x}^{\alpha DD} & -\frac{1}{2}T_{\xi\to x}^{ND} \\ \frac{1}{2}\tilde{T}_{\xi\to x}^{\alpha * DN} & R_{\xi\to x}^{NN} \end{bmatrix} \begin{bmatrix} g_\nu^D \\ g^N \end{bmatrix} = - \begin{bmatrix} \int_\Omega \gamma_\xi G_\xi^\alpha|_{\Gamma_D}(x) f(x) dx \\ \int_\Omega \mathcal{B}_{\nu_\xi}^\alpha G_\xi^\alpha|_{\Gamma_D}(x) f(x) dx \end{bmatrix}$$

$$+ \begin{bmatrix} -S_{\xi\to x}^{\alpha ND} & \frac{1}{2}(I_{\xi\to x}^{DD} + T_{\xi\to x}^{\alpha DD}) \\ \frac{1}{2}(I_{\xi\to x}^{NN} + \tilde{T}_{\xi\to x}^{\alpha * NN}) & -R_{\xi\to x}^{\alpha NN} \end{bmatrix} \begin{bmatrix} g^D \\ g_\nu^N \end{bmatrix}.$$

Lemma 2. *For a given association of a Lipschitz domain and constitutive functions parameters with a given source distribution, the Calderón operator with non null source is as a restriction which the Cauchy data must satisfy in order to be a consistent data with boundary value problems.*

The boundary integral equation methodology usually uses this Calderon's operator property to complete Cauchy data at the boundary but don't explores the consistence relation between Cauchy data and the model parameters, as we are done in this work.

2.5 The inverse problem

The appropriated formulation to the inverse parameter problem utilizes an domain Ω which is Lipschitz, that is, a domain with a boundary that can be locally modelled as the graph of a Lipschitz function, that is, a Holder continuous $C^{0,1}$ function. This is important since direct problems are frequently solved with numerical methods such as finite elements method or collocations methods, and engineering domains can be polygonal. In order to introduce a compact notation for the inverse problem, let $F_\alpha = [f_\alpha, ..., f_\alpha] \in (L^2(\Omega))^{m\times N_p}$ be the source and $(H, H_\nu) \in (H^{\frac{1}{2}}(\partial\Omega) \times (H^{-\frac{1}{2}}(\partial\Omega)))^{m\times N_p}$ the Cauchy data for N_p problems based on the $m-$fields model. Note that problems with different Cauchy data share the same source.

The inverse boundary value problem for parameter determination investigated here is: To find $(U, \alpha) \in H^1(\Omega)^{m\times N_p} \times \mathbf{R}^{N_a}$ such that

$$P_{F_\alpha, H, H_\nu}^\alpha \begin{cases} \mathcal{L}_\alpha U = F_\alpha & \text{if } x \in \Omega; \\ \gamma[U] = H & \text{if } x \in \partial\Omega; \\ \mathcal{B}_\nu[U] = H_\nu & \text{if } x \in \partial\Omega; \end{cases} \tag{2.19}$$

Here γ and $\mathcal{B}_\nu$ are the already defined traces for the second order elliptic problem. The coefficients of the strongly elliptic operator $\mathcal{L}_\alpha$ and the source depend on the parameters α.

2.6 The Parameters to Cauchy Data Implicit Function

There are some question in the parameter determination problem about how Cauchy data (H, H_ν) are related with the constitutive parameters α that are undetermined in the constitutive functions of the operator $\mathcal{L}_\alpha$ and the source f_α.

The first one is if there exist a general functional equation

$$\mathcal{C}^\alpha(H, H_\nu) = 0$$

which could solve our problem with in some operational set of numerical schemes, or, at least, that conduct us to a good framework to analysis? If this functional equation exist, what are its properties? What are the consequences of incorrect values on the parameters in it? Note that in an algorithm for parameter determination, we will necessarily need to trial parameters values different from the the appropriated ones. What are the consequence of incorrect values of the Cauchy Data? Cauchy data are supposed here to be overprescrited in order to compensate missing information on the parameters. So, the best functional will be the Calderon's operator.

By using the same properties the Calderon's operator that structures the Boundary integral formulation of elliptic systems, we will develop a methodology for the parameters determination problem by solving only direct problems such as (1.6) and the respective variational formulation (1.7) in an optimization context.

3 Complementary Problems

3.1 Complementary problems with normal boundary conditions

The Linear Boundary Value Problem in Ω can be a uniformly strongly elliptic system of partial differentials equations together with a system of **tangential differentials operators**, $\mathcal{R}$. In order to have **normal boundary conditions**, it is **fundamental** that there exists **complementary tangential boundary operators**, $\mathcal{S}$, such that the square matrix of tangential differentials operators, $\mathcal{M} := (\mathcal{R}, \mathcal{S})^T$, admits an inverse, $\mathcal{N} := \mathcal{M}^{-1}$, which is a matrix of tangential differential operators.

Definition 3. *The boundary value problem is called a regular elliptic boundary value problem is elliptic and the boundary conditions are normal and satisfy the Lopatinski-Shapiro condition.*

To the Lopatinski-Shapiro condition condition be satisfied, at each boundary point a special initial value problem in coordinate normal to the surface boundary at that point must be satisfied with trivial solution. For more information consult [9] Regular elliptic problems are the main class of problems used in the mathematical modelling of engineering problems. Complementary problems are defined in the sense that they satisfies the Lopatinski Shapiro condition and for a given set of Cauchy data they are expected to generates solutions fields equals in their respective associated complementary problems.

3.2 Complementary Cauchy data associated with Lipschitz boundary dissection

Let's consider the splitting of the Cauchy boundary data following some Lipschitz boundary dissection $\Gamma = \Gamma^{(1)} \cup \Pi \cup \Gamma^{(2)}$

$$H^{(1)} = \gamma[U]|_{\Gamma^{(1)}} \ ; \ H^{(2)} = \gamma[U]|_{\Gamma^{(2)}} \ ,$$

$$H_\nu^{(1)} = \mathcal{B}_\nu^\alpha[U]|_{\Gamma^{(1)}} \text{ and } H_\nu^{(2)} I = \mathcal{B}_\nu^\alpha[U]|_{\Gamma^{(2)}}$$

and solve with some guess of parameters values $\alpha = \alpha^{(0)}$ the $2 \times NP$ mixed boundary values problems

$$P^{\alpha^{(0)}}_{F_{\alpha^{(0)}}, H^{(1)}, H_\nu^{(2)}} \text{ and } P^{\alpha^{(0)}}_{F_{\alpha^{(0)}}, H^{(2)}, H_\nu^{(1)}}.$$

Let $U^{(1)}_{\alpha^{(0)}}$ and $U^{(2)}_{\alpha^{(0)}}$ be the its respective solutions. We immediately see that if the guess parameters induces a constitutive field different of the correct field near the boundary, some difference in the co-normal traces will be expected and consequent discrepancy in values of the fields of solution of the complementary problems associated. Meanwhile, what will be the situation if the constitutive fields coincides near the boundary, but are unequal in the interior of Ω. We will shown that even in this situation, a difference in the fields of solutions of complementary problems propagates to the interior and induces an *internal discrepancy field*. For the given Cauchy data and Lipschitz boundary dissection, this discrepancy only will disappears when the parameters in constitutive equations and sources are the correct ones. Of course it is induced by the different continuation to the interior of Ω due to incorrect parameters value.

3.3 Complementary problems on Lipschitz domains

Let us define the Complementary problems associated with regular normal elliptic problems.

Definition 4. *Let us consider two mixed boundary value problems* $P^\alpha_{f^{(1)}, g^{(1)}, g_\nu^{(1)}}$ *and* $P^\alpha_{f^{(1)}, g^{(2)}, g_\nu^{(2)}}$ *defined on the same Lipschitz domain* Ω *with* ***boundary dissection*** $\partial\Omega = \Gamma_D^{(1)} \cup \Pi \cup \Gamma_N^{(1)}$ *indexed as* L_d.

We say that these problems are complementary if the share the same source, $f_\alpha^{(1)} = f_\alpha^{(2)}$, *they interchange Dirichlet and Neumann boundaries,* $\Gamma_D^{(2)} = \Gamma_N^{(1)}$, $\Gamma_N^{(2)} = \Gamma_D^{(1)}$ *and there exist a Cauchy data* (g^α, g_ν^α) *compatible with the respective non null source Calderon's operator such that*

$$g^{(1)} = g^\alpha \chi_{\Gamma_D^{(1)}} \text{ and } g^{(2)} = g \chi_{\Gamma_D^{(2)}}. \tag{3.1}$$

$$g_\nu^{(1)} = g_\nu^\alpha \chi_{\Gamma_N^{(1)}} \text{ and } g_\nu^{(2)} = g_\nu^\alpha \chi_{\Gamma_N^{(2)}}.$$

where χ_Γ *is the characteristic function for set* Γ.

Theorem 1. *Suppose that two mixed boundary value problems* $P_{f_\alpha^{(1)}, g^{(1)}, g_\nu^{(1)}}$ *and* $P_{f_\alpha^{(2)}, g^{(2)}, g_\nu^{(2)}}$ *has respectively solutions* $u^{(1)}$ *and* $u^{(2)}$. *If they are complementary, then*

$$u^{(1)} = u^{(2)}.$$

Proof: Note that Γ is compact and

$$\Gamma = \Gamma_D^{(1)} \cup \Pi \cup \Gamma_N^{(1)} = \Gamma_D^{(2)} \cup \Pi \cup \Gamma_N^{(2)}.$$

With the characteristic function for $\Gamma_D^{(1)}$, $\chi_{\Gamma_D^{(1)}}$ in Γ, we can introduce the extended Dirichlet data function $g \in H^{\frac{1}{2}}(\Gamma)$ which is the extension of $g^{(1)} \in H^{\frac{1}{2}}(\Gamma_D^{(!)})$ to $H^{\frac{1}{2}}(\Gamma)$, with $g^{(1)} = g\chi_{\Gamma_D^{(1)}}$, and consequently, $g^{(2)} = g\chi_{\Gamma_N^{(2)}}$. Similar definitions can be done to treat the Neumann part of Cauchy data $g_\nu \in H^{-\frac{1}{2}}(\Gamma)$, $g_\nu^{(1)} = g_\nu \chi_{\Gamma_D^{(1)}}$. So,

$$g(x) = g^{(1)}(x)\chi_{\Gamma_D^{(1)}}(x) + g^{(2)}(x)\chi_{\Gamma_N^{(1)}}(x) = g^{(1)}(x)\chi_{\Gamma_N^{(1)}}(x) + g^{(2)}(x)\chi_{\Gamma_D^{(2)}}(x)$$

and

$$g_\nu(x) = g_\nu^{(1)}(x)\chi_{\Gamma_D^{(1)}}(x) + g_\nu^{(2)}(x)\chi_{\Gamma_N^{(1)}}(x) = g_\nu^{(1)}(x)\chi_{\Gamma_N^{(2)}}(x) + g_\nu^{(2)}(x)\chi_{\Gamma_D^{(2)}}(x).$$

Denoting $f_\alpha = f^{(1)} = f^{(2)}$, the solution will be, via boundary integral equation method,

$$u(\xi) = \int_\Omega G_\xi^\alpha(x) f_\alpha(x)dx - DL_\xi^\alpha[g](x) + SL_\xi^\alpha[g_\nu](x), \xi \in \Omega \tag{3.2}$$

By taking the trace and the conormal trace of (3.2), we see that it satisfies the Calderón's operators dissection equation(3.3). So, Cauchy data obtained by the extension formulates a unique problem with integral representation (3.2),which is the matrix equation for Calderon's operator with non null source in the context of Lipschitz boundary dissection:

$$\begin{bmatrix} g(\xi)|_{\Gamma_D} \\ g(\xi)|_{\Gamma_N} \\ g_\nu(\xi)|_{\Gamma_D} \\ g_\nu(\xi)|_{\Gamma_N} \end{bmatrix} = \begin{bmatrix} \gamma u(\xi)|_{\Gamma_D} \\ \gamma u(\xi)|_{\Gamma_N} \\ \mathcal{B}_\nu^\alpha u(\xi)|_{\Gamma_D} \\ \mathcal{B}_\nu^\alpha u(\xi)|_{\Gamma_N} \end{bmatrix} = \begin{bmatrix} \int_\Omega \gamma_\xi G_\xi^\alpha|_{\Gamma_D}(y) f_\alpha(y)dy \\ \int_\Omega \gamma_\xi G_\xi^\alpha|_{\Gamma_N}(y) f_\alpha(y)dy \\ \int_\Omega \mathcal{B}_{\nu_\xi}^\alpha G_\xi^\alpha|_{\Gamma_D}(y) f_\alpha(y)dy \\ \int_\Omega \mathcal{B}_{\nu_\xi}^\alpha G_\xi^\alpha|_{\Gamma_D}(y) f_\alpha(y)dy \end{bmatrix} + \tag{3.3}$$

$$\begin{bmatrix} \frac{1}{2}(I_{x\to\xi}^{DD} - T_{x\to\xi}^{\alpha DD}) & -T_{x\to\xi}^{\alpha ND} & S_{x\to\xi}^{\alpha DD} & S_{x\to\xi}^{\alpha ND} \\ -T_{x\to\xi}^{\alpha DN} & \frac{1}{2}(I_{x\to\xi}^{NN} - T_\xi^{\alpha NxN}) & S_{x\to\xi}^{\alpha DN} & S_{x\to\xi}^{\alpha NN} \\ R_{x\to\xi}^{\alpha DD} & R_{x\to\xi}^{\alpha ND} & \frac{1}{2}(I_{x\to\xi}^{DD} + \tilde{T}_{x\to\xi}^{\alpha * DD}) & \tilde{T}_{x\to\xi}^{\alpha * ND} \\ R_{x\to\xi}^{\alpha DN} & R_{x\to\xi}^{\alpha NN} & \tilde{T}_{x\to\xi}^{\alpha * DN} & \frac{1}{2}(I_{x\to\xi}^{NN} + \tilde{T}_{x\to\xi}^{\alpha * NN}) \end{bmatrix} \begin{bmatrix} \gamma u(x)|_{\Gamma_D} \\ \gamma u(x)|_{\Gamma_N} \\ \mathcal{B}_\nu^\alpha u(x)|_{\Gamma_D} \\ \mathcal{B}_\nu^\alpha u(x)|_{\Gamma_N} \end{bmatrix}$$

Remark 2. *If Cauchy data are compatible with a Calderon's operator involving a different set of constitutive and source parameters, then Cauchy data, consistent with a set, α, of parameters, will not be consistent with this new set, say, $\alpha^{(0)}$ and also the calculated complementary data will extended different fields associated with the volumetric potential, with the single layer and double layer potential fields inside Ω. As consequence of this, a discrepancy between the two fields of solutions of complementary problems*

$$D^{(1,2)}_{\alpha^{(0)}, L_d, Cauchy^\alpha} := u^{(1)}_{\alpha^{(0)}, L_d, Cauchy^\alpha} - u^{(2)}_{\alpha^{(0)}, L_d, Cauchy^\alpha}$$

will take place to indicate that we have introduced a wrong parameters values. Note that we subscript with the symbols $\alpha^{(0)}$ to indicate a trial with a wrong parameters value, L_d to indicate an specific Lipschitz dissection of Cauchy data and $Cauchy^\alpha$ to indicate that Cauchy data are consistent we the exact parameters value α.

4 Discrepancy Field in the Diffusion-Absorption Elliptic Model

One of the simplest model with non uniform materials properties in which discrepancy fields can be investigated is the diffusion ($c_\alpha(x)$) and absorption ($a_\alpha(x)$ model problem:

$$\begin{array}{ll} \mathcal{L}_\alpha u(x) = -\nabla \cdot c_\alpha(x)\nabla u(x) + a_\alpha(x)u(x) & \text{if } x \in \Omega; \\ \gamma[u](x) = u(x) & \text{if } x \in \partial\Omega_D; \\ \mathcal{B}_\nu^\alpha[u](x) = c_\alpha(x)\nabla u(x) & \text{if } x \in \partial\Omega_N; \end{array} \tag{4.1}$$

4.1 Dirichlet functional

The main properties of the system are given by the functional

$$\Phi_\alpha(u, v) := \int_\Omega [c_\alpha(x)\nabla u(x)\nabla v(x) + a_\alpha(x)u(x)v(x)]dx \tag{4.2}$$

with has remarcable properties such as, if $u = v$ are functions in a normed space, then $\Phi_\alpha(u, u)$ is the energy norm. If $u = \phi_i$ and $v = \phi_j$, $i, j = 1, 2, 3, ...$ are finite elements basis in a Galerkin approximation, then $\Phi_\alpha(u,u)$ is the sum of stiffness and absorption matrix. If $u = v = \phi_i$, $i = 1,2,3,...$ are orthonormal eigenfunctions in the spectral problem for this model, then $\lambda_i := \Phi_\alpha(\phi_i,\phi_i)$ are the respective eigenvalue. The rate of decay of the sequence $\{\mu_i := 1/\Phi_\alpha(\phi_i,\phi_i), i = 1,2,...\}$ gives information about the ill-conditioning of this system of the inverse coefficients problems associated with this model. Moderately ill-posed for polynomial decay and severely ill posed for exponential decay.

4.2 Weak mixed problem

The weak solution for mixed problems with this model is implemented with the First Green Identity:

$$\Phi_\alpha(u,v) = \int_\Omega \mathcal{L}_\alpha u(x)v(x)dx + \int_{\partial\Omega} \mathcal{B}_\nu^\alpha[u](x)\gamma[v](x) \text{ for all } v \in H^1(\Omega) \tag{4.3}$$

The variational interpretation of the First Green Identity gives a weak formulation, problem $W^\alpha_{f_\alpha,g^D,g^N_\nu}$, for the mixed problem (1.2): Find $(u,\lambda) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega_N)$ where λ is a conormal trace of $H^1(\Omega)$.

$$\begin{cases} \Phi_\alpha(u,v) - \langle\gamma[v],\lambda\rangle_{\partial\Omega_D} = \langle f_\alpha, v\rangle_\Omega + \langle g^N, \gamma[v]\rangle_{\partial\Omega_N} \\ \langle\gamma[u],\mu\rangle = \langle g^D,\mu\rangle_{\partial\Omega_D} \\ \forall(v,\mu) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega_N). \end{cases} \tag{4.4}$$

Note that $\langle\gamma[v],\lambda\rangle_{\partial\Omega} = \int_{\partial\Omega}\gamma[v]\lambda ds_x = \int_\Omega \gamma^*[\lambda]vdx = \langle\gamma^*[\lambda],v\rangle_\Omega$ defines the an extension operator $\gamma^*[.] : H^{-\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$. The Lagrangian functional is for this mixed problem is

$$\mathcal{A}_\alpha(v,\lambda) := \frac{1}{2}\Phi_\alpha(v,v) - \langle\gamma[v],\lambda\rangle_{\partial\Omega_D} - \langle f,v\rangle_\Omega + \langle g_D,\lambda\rangle_{\partial\Omega_D} - \langle\gamma[v],g_N\rangle_{\partial\Omega_N}$$

and the critical point variational formulation is stated by

Theorem 2 (Lagrangian Functional Critical Point). *The pair of fields $(u,\lambda) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Omega)$ is solution of the mixed problem $W^\alpha_{f_\alpha,g^D,g^N_\nu} \Leftrightarrow \mathcal{A}_\alpha(u,\mu) \le \mathcal{A}_\alpha(u,\lambda) \le \mathcal{A}_\alpha(v,\lambda)$ for all $(v,\mu) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Omega)$.*

The unique solvability of the saddle point problem (4.4) is assured by the Lemma

Lemma 3 (Stability condition). *The stability condition*

$$c_S|||\lambda|||_{H^{-\frac{1}{2}}(\partial\Omega_D)} \le \sup_{0\ne v\in H^1(\Omega)} \frac{(\gamma_0 v,\lambda)_{\partial\Omega_D}}{|||v|||_{H^1(\Omega)}}$$

is satisfied for all $\lambda \in H^{-\frac{1}{2}}(\partial\Omega_D)$.

Demostration of the theorem (2) and lemma (3) can be found in [8].

5 Finite Elements Formulation

The weak formulation for mixed problems can be computationally implemented with linear Lagrangian finite elements. The linear system to be solved is given by the following problem: To find $(U_\alpha, \Lambda_{dir}) \in \mathbf{R}^{N_v\times Np} \times \mathbf{R}^{N_{dir}\times Np}$ such that

$$\begin{cases} (K_\alpha + A_\alpha)U_\alpha - Tr^T_{dir}\Lambda_{dir} = F_\alpha + Tr^T_{neu}G_{neu} \\ Tr_{dir}U_\alpha = G_{dir} \end{cases}$$

where $size(K_\alpha) = size(A_\alpha) = [N_v, N_v]$, $size(F_\alpha) = [N_v, N_p]$, $size(G_{dir}) = [N_{dir}, N_p]$, $size(G_{neu}) = [N_{neu}, N_p]$, $size(Tr_{dir}) = [N_{dir}, N_v]$ and $size(Tr_{neu}) = [N_{neu}, N_v]$.

Here N_v , N_{dir} , N_{neu} and N_p are respectively the number of vertices on $\Omega \cup \partial\Omega$, $\partial\Omega_N$, and the number of problems with the same parameters α values and same Lipschitz boundary dissection. We also note that in order of facilitates the iterative calculations with guess parameters, the assemble of matrices must minimizes the computational cost of recalculations with trial values of the parameters.

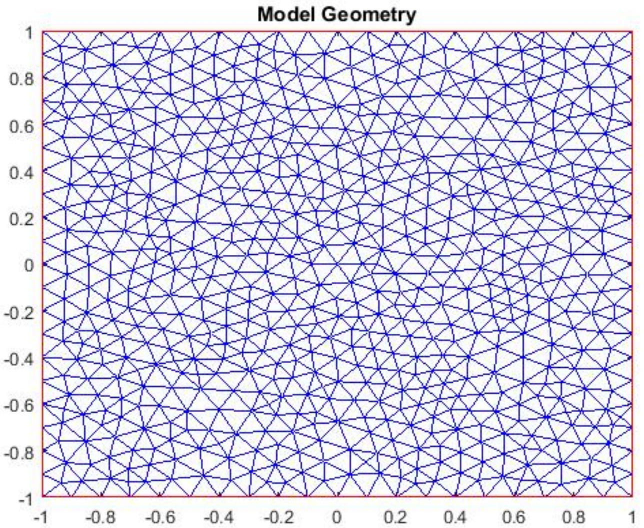

Fig. 1. Typical finite elements mesh used in the model

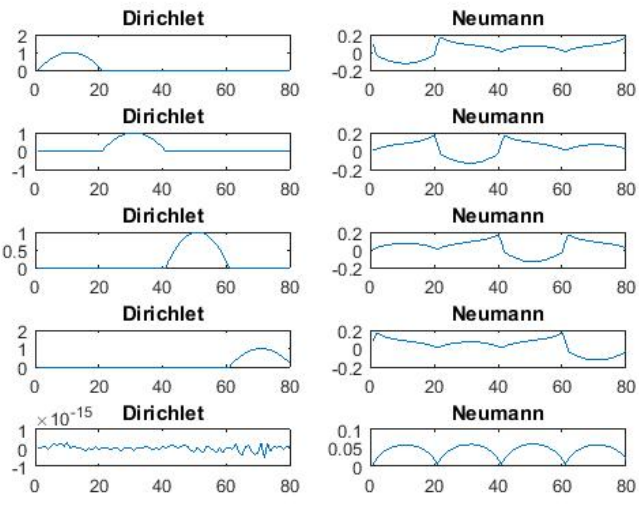

Fig. 2. Typical synthetic Cauchy data calculated with Lagrange multiplier

Table 1. Lagrange multipliers $L^2(\Omega)$ errors and order of convergence estimation with mesh refinement

L	N_Ω	N_Γ	$\|\|c\frac{\partial u_m}{\partial \nu} - \lambda_h\|\|^1_{L^2(\Gamma)}$	coc^1	$\|\|c\frac{\partial u_m}{\partial \nu} - \lambda_h\|\|^2_{L^2(\Gamma)}$	coc^2
2	29	16	6.5485	$---$	2.0245	$---$
3	92	32	1.4720	2.1534	0.4083	2.3099
4	322	64	0.3547	2.0531	0.0765	2.4116
1	1232	128	0.0872	2.0235	0.0139	2.4625
6	4833	256	0.0217	2.0100	0.0025	2.4897
7	19090	512	0.0054	2.0041	0.0004	2.4998
8	74562	1024	0.0013	2.0016	0.0001	2.5018

We have implement the finite elements method with Lagrange multiplier to facilitate calculations involving the solution of the mixed boundary problems of kind (4.4) in a more straightforward way. Of course, the use of pde solvers such as the Matlab pde tool box or others similar solvers can also be adopted. Typical finite elements mesh used for produces synthetic Cauchy data calculated with Lagrange multiplier are shown in Figs. 1-2, respectively. The Lagrange element shape adopted is linear. We have check with computer experiment the possibility of violate the sufficient condition of adoption of a Lagrange multiplier mesh with at least double size of the solution field mesh and obtain a positive result for adoption of the same mesh for the two calculations.

We made $L^2(\Omega)$ estimate by supposing that the Matlab PDE toolbox calculations of the conormal trace, $c\frac{\partial u_m}{\partial \nu}$, are appropriated reference. In check of order the convergence, denoted by coc, we use [8] the Aubin-Nitche trick

$$coc := \frac{log(\|\|c\frac{\partial u_m}{\partial \nu} - \lambda_{h_l}\|\|_{L^2(\Omega)}) - log(\|\|c\frac{\partial u_m}{\partial \nu} - \lambda_{h_{l+1}}\|\|_{L^2(\Omega)})}{log(h_l) - log(h_{l+1})}. \quad (5.1)$$

The typical coefficients parameters values adopted in the model are the following. The conductivity is calculated with the function $c = 1 + exp(-C * ((x - x_c)^2 + (y - y_c)^2)))$, where C, x_c, y_c are the conductivity parameters. The absorption coefficients is constant $a = 1$. The source is also constant $f = 10$. The Dirichlet boundary data can be zero or calculated by a function like $u = u_{max} * (1 - s) * (1 + s)$, where s is the variable on the edge and its maximum value is $u_{max} = 10$. In Table 1 we shown a check of $L^2(\Omega)$ error and order of convergence for those typical values. The superscript 1 and 2, respectively, indicates the case of non null Dirichlet data given by given function in one of the edges and zero in all others edges, and the case of null Dirichlet data in all edges.

5.1 Existence of discrepancy fields between complementary solutions

The numerical determination of the internal discrepancy field for diffusion absorption model can be stated in the following way: since for a given association of a Lipschitz domain with a source distribution, the Calderon's operator with non homogeneous source is a restriction which the Cauchy data must satisfy in order to be a consistent data with boundary value problems. If the inverse problem $P^{\alpha}_{F_\alpha,H,H_\nu}$ is solved with trial parameters values $\alpha^{(0)} \neq \alpha$, which is the exact value, the associated Calderon's operator will present a gap. Then the Complementary solutions associated with Cauchy data will misfit the calculated values, that is presents a internal discrepancy between calculated solutions.

In the next Fig. 3 we shown an example of the Discrepancy Field that will depend on Lipschitz boundary Dissection. The supposed exact parameters are $[C, x_c, y_c, a, f] = [6, .1, .2, 1, 1]$ and the trial values are $[3, .5, -.5, 3, 2]$.

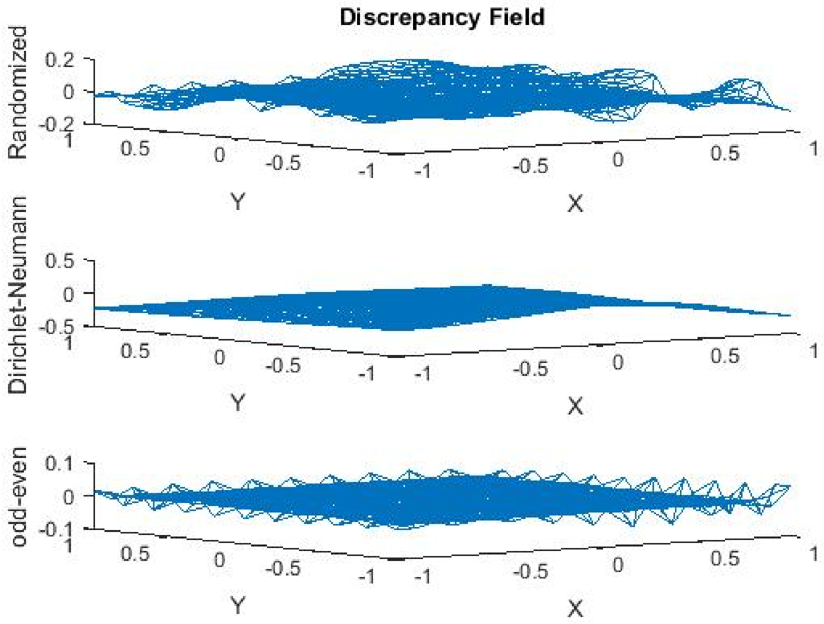

Fig. 3. Numerical discrepancy field calculation with parameters values different from exact ones

5.2 The variational method for the discrepancy field

The discrepancy field is a difference between two solution of the operator equation $\mathcal{L}_\alpha$ with the same source f_α. An immediate consequence of this is that it is an $\mathcal{L}_\alpha$-harmonic function. That is:

$$D^{(1,2)}_{\alpha^{(0)},L_d,Cauchy^\alpha} \in H^1_{\mathcal{L}_\alpha}(\Omega) := \{v \in H^1(\Omega) | \mathcal{L}_\alpha v = 0\}$$

Also, the First Green's Identity states

$$\Phi_\alpha(D^{(1,2)}_{\alpha^{(0)},L_d,Cauchy^\alpha}, v) = \int_{\partial\Omega} \mathcal{B}^\alpha_\nu[D^{(1,2)}_{\alpha^{(0)},L_d,Cauchy^\alpha}](x)\gamma[v](x) \text{ for all } v \in H^1(\Omega) \quad (5.2)$$

Proposition 1 (Null Set)**.** *For a Lipschitz Boundary Dissection on $\partial\Omega$ indexed as $L_d$and the Cauchy Data consistent with a set of parameters α and discrepancy $D^{12}_{\alpha_0}$ for complementary problems calculated with trial parameters α_0 , the Dirichlet Functional*

$$\Phi_\alpha(u^{(1)}_{\alpha_0,L_d,Cauchy^\alpha} - u^{(2)}_{\alpha_0,L_d,Cauchy^\alpha}, v) = 0$$

for all test function $v \in H^1_0(\Omega)$. The numerical evidence of this proposition can be seen in the next Fig. 4. In these calculation, until index 80 we have boundary points and non null values of Dirichlet functional. For the other interior points the functional is null.

This results evidence the interior character of the discrepancy fields.

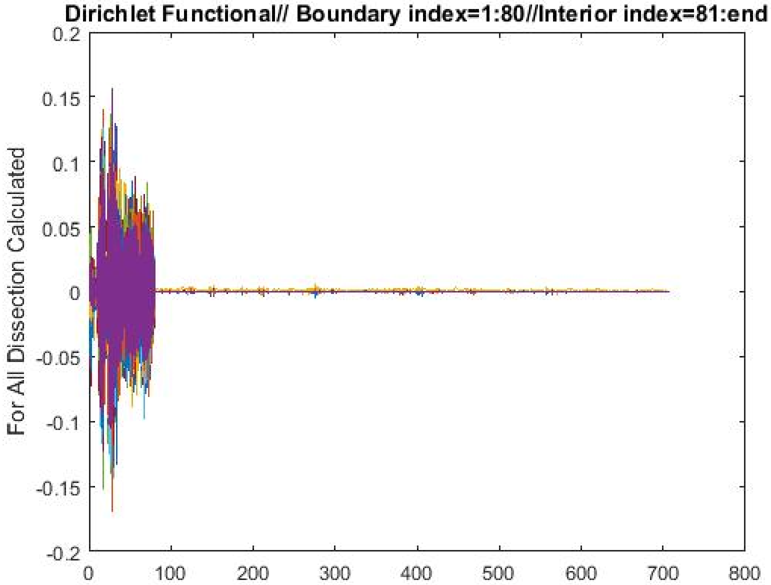

Fig. 4. Dirichelet Functional at boundary and interior test functions

5.3 Optimization problem based on the discrepancy fields

The inverse parameters Problem (2.19) can now be posed with the following optimization problem:

Problem 1. *In the guess set of parameters $\alpha^{(0)}$ in the interval$\{[\alpha_1, \alpha_2] \subset \mathbf{R}^{N_\alpha}\}$, to find the set of parameters α that minimizes some distance between complementary solutions*

$$u^{(1)}_{\alpha, Ld, Cauchy^\alpha} \text{ and } u^{(2)}_{\alpha, Ld, Cauchy^\alpha}$$

for all parameters consistent Cauchy data and all respective Lipschitz dissected solutions.

5.4 Distance based on the discrepancy

Based on Theorem of Complementary Solutions we create some discrepancy function that measures observed differences for guess value of the parameters. Norms in the solution space for the direct problems can be adopted, that is,

$$d_{\alpha^{(0)}, L_d, Cauchy} = ||u^{(1)}_{\alpha^{(0)}, L_d, Cauchy} - u^{(2)}_{\alpha^{(0)}, L_d, Cauchy}||_V, \tag{5.3}$$

where V can be some norm or any other measure such as Kullback Leibler divergence or Bregmann distance. The simplest will be to choose the Chebyshev distance in the context of Least Squares method. So

$$d^{\infty}_{\alpha^{(0)}, L_d, Cauchy} = \max_{Ld, N_v} \{u^{(1)}_{\alpha^{(0)}, L_d, Cauchy^\alpha} - u^{(2)}_{\alpha^{(0)}, L_d, Cauchy^\alpha}\}, x \in \Omega)$$

is appropriated since finite elements solutions are usually continuous. Numerical experiment related with the distance based discrepancy and Nelder-Mead optimization can be found in [10].

5.5 The least squares model

Let α be the correct unknown parameter vector. Let $\alpha^{(0)}$ some parameter guess. The first order expansion in α of the exact discrepancy field

$$u^{(1)}_{j,\alpha,L_d} - u^{(2)}_{j,\alpha,L_d} = 0 \tag{5.4}$$

suggest the following approximation system to be solved in the least squares sense

$$u^{(1)}_{j,\alpha^{(0)},L_d} - u^{(2)}_{j,\alpha^{(0)},L_d} + \sum_{k=1}^{N_\alpha} \frac{\partial}{\alpha_k}(u^{(1)}_{j,\alpha,L_d} - u^{(2)}_{j,\alpha,L_d})|_{\alpha^{(0)}} \Delta\alpha_k = 0 \tag{5.5}$$

for all nodal values at vertices $j = 1, ..., N_v$, all Lipschitz dissections $L_d = 1, ..., N_{L_d}$ and all Cauchy data compatible with the exact parameters value. The linear system inversion obviously must, if necessary, be stabilized with an appropriated choose of a regularization methodology. The model based on diffusion-absorption problem in the bi-dimensional square geometry shown in Fig. 1 can be programmed to solve the system (5.5).

The exact conductivity parameters used to generates the synthetic Cauchy data are $[C, x_c, y_c, a, f] = [6, 0.1, 0.2, 1, 1]$. Least squares errors on the vector of parameters with the conductivity, absorption and source parameters starting with the following set of parameters trial $[C_0, x_0, y_0, a_0, f_0] = [3, .5, -.5, 3, 2]$ They are reconstructed with minimizing of least squares error. Fig. 5 shown the errors evolution with iteration. Finally, the least-squares reconstruction is shown in Fig. 6. The influence of the Lipschitz dissection investigation will not be presented in this work.

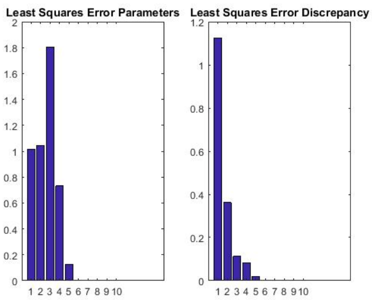

Fig. 5. Convergence of parameters to exact values under the least squares method

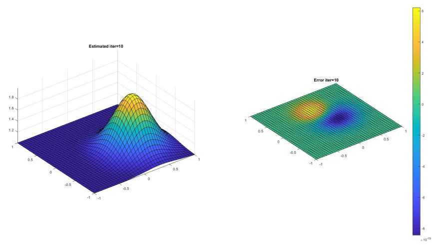

Fig. 6. Least squares reconstruction of conductivity

6 Conclusions

In this work we present a methodology based on over prescription of Cauchy data with Lipschitz Boundary Dissection for elliptic systems parameters determination. A finite elements formulation for solution of Multiple Complementary Direct Mixed Problems with wrong values of trials parameters is presented. We explore the concept of Complementary Solutions and the existence of Discrepancy Fields for trials with wrong parameters values on coefficients and sources functions. Based on Least Squares and L^∞ norm of Discrepancy Fields, we presents numericals experiments of parameters determination. Based on the results the existence of discrepancy fields for complementary mixed problems is put in evidence and so its dependence on wrong values of constitutives parameters can be use for correct parameters values consistents with given Cauchy data.

Acknowledgements

This work is supported by Brazilian Agencies CNPq and CAPES.

Competing Interests

Author has declared that no competing interests exist.

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Available:https://doi.org/10.1155/2013/631950

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