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Test of Unit Root for Bounded AR (2) Model

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Authors' contributions

This work was carried out in collaboration among all authors. Author MAFA designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Authors SMES and AAES managed the analyses of the study. Author AAES managed the literature searches. All authors read and approved the final manuscript.

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Abstract

In this paper, the test of unit root for bounded AR (2) model with constant term and dependent errors has been derived. Asymptotic distributions of OLS estimators and $t-type$ statistics under different tests of hypotheses have been derived. A simulation study has been established to compare between different tests of the unit root. Mean squared error (MSE) and Thiel's inequality coefficient (Thiel's U) have been considered as criteria of comparison.

Keywords: Bounded AR (2) model; asymptotic distributions; OLS estimators; $t-type$ statistics; mean squared error; Thiel's inequality coefficient.

1 Introduction

Many unit root tests have been developed for testing the null hypothesis of a unit root against the alternative of stationarity, the tests for unit roots in AR (1) processes were first proposed and investigated by Dickey

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and Fuller [1,2] but these unit root tests are proposed to unbounded time series in case of independent error terms.

Cavaliere [3] tested the presence of unknown boundaries which constrain the sample path to lie within a closed interval that is in the framework of integrated processes of AR (1) model with a unit root or random walk model (with and without linear trend) and in (2002) he introduced the logged nominal exchange rates $\{y_t\}$ that change in time accordingly to a first-order integrated process, $I\,(1)$ within the framework of non-managed flexible exchange rates. In (2005), Cavaliere [4] developed an asymptotic theory for integrated and near-integrated time series whose range is constrained in some ways. Such a framework arises when integration and cointegration analysis are applied to persistent series which are bounded either by construction or because they are subject to control.

Cavaliere and Xu [5] defined bounded process as time series x_t with (fixed) bounds at $\underline{b}\,,\overline{b}\,;\,\underline{b}<\overline{b}$, is a stochastic process satisfying $x_t \in [\,\underline{b}\,,\overline{b}\,]$ for all *t*.

Carrion and Gadea (2013) showed that the use of generalized least squares (GLS) detrending procedures leads to important empirical power gains compared to ordinary least squares (OLS) detrending method when testing the null hypothesis of unit root for bounded processes. In (2015), they discussed the unit root testing when the range of the time series is bounded considering the presence of multiple structural breaks. But they all concentrated on the model of bounded AR (1) with constant or without constant under various assumptions for the error terms, and in this paper the concentration will be on the bounded AR (2) with constant model in case of dependent errors.

2 Test of Unit Root for Bounded AR (2) Model with constant Term in Case of Dependent Errors

The bounded second order autoregressive AR (2) model takes the form:

$$y_t = \alpha + \rho_1\, y_{t-1} + \rho_2\, y_{t-2} + u_t\,, \qquad t = 1,\ldots,T, \tag{1}$$

where y_t is bounded time series with fixed bounds with lower bound at $\underline{b}$ and upper bound at $\overline{b}$, $y_t \in [\underline{b},\overline{b}]$, and $\underline{b} = \underline{c}\,T^{1/2}[1-\phi_1]^{-1}$, $\overline{b} = \overline{c}\,T^{1/2}[1-\phi_1]^{-1}$ and T is the sample size, $\underline{c}\,,\overline{c} \in R/\{0\}$ and $\underline{c} < \overline{c}$, $\phi_1 = \{\pm 0.1, \pm 0.2, \ldots, \pm 0.9\}$, $y_0 = y_{-1} = 0$, u_t are dependent error terms which achieved Beveridge-Nelson Decomposition, ρ_1 and ρ_2 are the autoregressive coefficients and α is the constant term.

2.1 Asymptotic distributions of OLS estimators under different tests of hypothesis

Concepts of relative magnitude or order of magnitude are useful in investigating limiting behavior of random variables, where if $h\,(x)$ and $g\,(x)$ are two real functions that have a common domain $D \subset R$, and if the following relationship is exists for any positive constant $k\;(k\, >\, 0)$

$$\lim_{x \to x_0} \left| \frac{h(x)}{g(x)} \right| \leq k \quad , \quad x \in (D - x_0)$$

Where, $h(x) = O(g(x))$. (2)

Schatzman [6]

If $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ an $m \times n$ matrix with $r^* = rank(A)$ where B is $r^* \times r^*$ and invertible then the generalized inverse G for a given singular matrix A can be obtained as follows:

$$G = \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{3}$$

And if an equation represented as:

$$Ax = h \,, \; x \in R$$

Where, x is a vector or a matrix of unknown elements, h is vector or a matrix that has the same order as the product of $A\,x$ **.** So, to obtain the forms of unknown elements of x the following equation is need to be used:

$$x = Gh + (I - GA)z \,, \; z \in R \tag{4}$$

Where I is an identity matrix, z is a vector or a matrix of real numbers and G is the generalized inverse of the matrix A that satisfied $A\,GA = A$ **.** Sawyer [7]

If y_t is a pure random walk without drift as $y_t = y_{t-1} + u_t$, where $y_0 = y_{-1} = 0$, u_t are dependent error terms, and assume that u_t is defined as follows:

$$u_t = \phi_1 u_{t-1} + e_t = \psi(L)\, e_t = \sum_{j=0}^{\infty} \psi_j \, e_{t-j} \;, \left|\phi_1\right| < 1 \;, t = 1, 2, \ldots, T \tag{5}$$

Where:

$$\left.\begin{array}{l} E(e_t) = 0 \quad \text{for all } t \\ E(e_t\, e_s) = \begin{cases} \sigma^2 & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases} \\ \sum_{j=0}^{\infty} |j| \, \psi_j < \infty \end{array}\right\} \tag{6}$$

Then the following relationship exists:

$$\sum_{s=1}^{t} u_s = \psi(1) \sum_{s=1}^{t} e_s + \eta_t - \eta_0 \quad , t = 1, 2, ..., T \tag{7}$$

$$\left.\begin{array}{l} \psi(1) = \sum_{j=0}^{\infty} \psi_j \\ \eta_t = \sum_{j=0}^{\infty} a_j e_{t-j} \\ a_j = -(\psi_{j+1} + \psi_{j+2} + \psi_{j+3} + \ldots) = -\sum_{i=1}^{\infty} \psi_{j+i} \ , \quad \sum_{j=0}^{\infty} \left| a_j \right| < \infty \\ \eta_0 = -(\psi_1 + \psi_2 + \psi_3 + \ldots) e_0 - (\psi_2 + \psi_3 + \psi_4 + \ldots) e_{-1} \\ \qquad - (\psi_3 + \psi_4 + \psi_5 + \ldots) e_{-2} + \ldots\ldots \end{array}\right\}$$

By defining the following quantities:

$$\left.\begin{array}{l} \gamma_j = E(u_t u_{t-j}) = \sigma^2 \sum_{s=0}^{\infty} \psi_s \psi_{s+j} \quad , j = 0, 1, 2, ... \\ \lambda = \sigma \sum_{j=0}^{\infty} \psi_j = \sigma \ \psi(1) \\ y_t = u_1 + u_2 + \ldots + u_t \quad , t = 1, 2, \ldots, T \end{array}\right\} \tag{8}$$

Then the following results are obtained:

$$\left.\begin{array}{l} 1) \ T^{-1} \sum_{t=1}^{T} u_t u_{t-j} \xrightarrow{p} \gamma_j \ , \ j = 0, 1, 2, \ldots \\ 2) \ T^{-1} \sum_{t=1}^{T} y_{t-1} u_t \xrightarrow{d} \frac{1}{2} \{ \lambda^2 [W(1)_{\underline{c}}^{\bar{c}}]^2 - \gamma_0 \} \\ 3) \ T^{-3/2} \sum_{t=1}^{T} y_{t-1} \xrightarrow{d} \lambda \int_0^1 W(r)_{\underline{c}}^{\bar{c}} dr \\ 4) \ T^{-2} \sum_{t=1}^{T} y_{t-1}^2 \xrightarrow{d} \lambda^2 \int_0^1 [W(r)_{\underline{c}}^{\bar{c}}]^2 dr \end{array}\right\} \tag{9}$$

Where, $W_{\underline{c}}^{\bar{c}}(r)$ is a Regulated Brownian Motion, when $r = 1$, then:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} u_t \xrightarrow{d} \sigma \psi(1) W_{\underline{c}}^{\bar{c}}(1) = \lambda W_{\underline{c}}^{\bar{c}}(1) \tag{10}$$

By using equation (2) the results for orders of convergence of estimators in these equations will be as follows:

$$\left.\begin{array}{l} 1) T = O_p(T) \\ 2) \ \sum_{t=1}^{T} u_t = O_p(T^{1/2}) \\ 3) \sum_{t=1}^{T} u_t u_{t-j} = O_p(T) \\ 4) \sum_{t=1}^{T} y_{t-1} u_t = O_p(T) \\ 5) \sum_{t=1}^{T} y_{t-1} = O_p(T^{3/2}) \\ 6) \sum_{t=1}^{T} y_{t-1}^2 = O_p(T^2) \end{array}\right\}. \tag{11}$$

Amer [8]

The **asymptotic distributions of OLS estimators** $\hat{\alpha}, \hat{\rho}_1$ and $\hat{\rho}_2$ for bounded AR (2) model that represented by equation (1) for testing the null hypothesis $H_0 : \alpha = 0, \rho_1 = 1, \rho_2 = 0$, (i.e. $y_t = y_{t-1} + u_t$) against the alternative hypothesis $H_a : \alpha \neq 0$, $|\rho_1| < 1$, $|\rho_2| < 1$, (i.e. $y_t = \alpha + \rho_1 y_{t-1} + \rho_2 y_{t-2} + u_t$) **will be** derived **as follows:**

Lemma (1): If y_t is a pure random walk without drift as $y_t = y_{t-1} + u_t$, where $y_0 = y_{-1} = 0$, u_t are dependent error terms that **achieved** the Beveridge-Nelson Decomposition as in equation (7) then as $T \to \infty$ the following results are obtained:

$$\left.\begin{array}{l} 1) T^{-1} \sum_{t=1}^{T} y_{t-2} u_t \xrightarrow{d} \frac{1}{2} \{ \lambda^2 [\bar{W}_{\underline{c}}^{c}(1)]^2 - \gamma_0 \} - \gamma_1 \\ 2) T^{-3/2} \sum_{t=1}^{T} y_{t-2} \xrightarrow{d} \lambda \int_0^1 \bar{W}_{\underline{c}}^{c}(r) dr \\ 3) T^{-2} \sum_{t=1}^{T} y_{t-2}^2 \xrightarrow{d} \lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2 dr \\ 4) T^{-2} \sum_{t=1}^{T} y_{t-1} y_{t-2} \xrightarrow{d} \lambda^2 \int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2 dr \end{array}\right\} \tag{12}$$

Where, $\gamma_0 = \sigma^2 \sum_{s=0}^{\infty} \psi_s^2$, $\gamma_1 = \sigma^2 \sum_{s=0}^{\infty} \psi_s \psi_{s+1}$ and $\lambda = \sigma \sum_{j=0}^{\infty} \psi_j = \sigma \psi(1)$.

Proof:

Part (1)

From the successive substituting of y_t then:

$$y_{t-2} = y_{t-1} - u_{t-1} \tag{13}$$

So,

$$T^{-1} \sum_{t=1}^{T} y_{t-2} u_t = T^{-1} \sum_{t=1}^{T} y_{t-1} u_t - T^{-1} \sum_{t=1}^{T} u_{t-1} u_t \tag{14}$$

By using equation (9) then:

$$\left.\begin{array}{l} 1) T^{-1} \sum_{t=1}^{T} y_{t-1} u_t \xrightarrow{d} \frac{1}{2} \{ \lambda^2 [\bar{W}_{\underline{c}}^{c}(1)]^2 - \gamma_0 \} \\ 2) T^{-1} \sum_{t=1}^{T} u_{t-1} u_t \xrightarrow{d} \gamma_1 \end{array}\right\} \tag{15}$$

Then, by substituting from equations (15) in (14) it can be concluded that:

$$T^{-1} \sum_{t=1}^{T} y_{t-2} u_t \xrightarrow{d} \frac{1}{2} \{ \lambda^2 [\bar{W}_{\underline{c}}^{c}(1)]^2 - \gamma_0 \} - \gamma_1$$

Part (2)

From equation (13);

$$T^{-3/2}\sum_{t=1}^{T} y_{t-2} = T^{-3/2}\sum_{t=1}^{T} y_{t-1} - T^{-3/2}\sum_{t=1}^{T} u_{t-1} \tag{16}$$

From equation (11) the order of convergence of $\sum_{t=1}^{T} u_{t-1} = O_P(T^{1/2})$ then:

$$T^{-3/2}\sum_{t=1}^{T} u_{t-1} \xrightarrow{d} 0 \tag{17}$$

By using equation (9) it can be concluded that:

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1} \xrightarrow{d} \lambda \int_0^1 W_{\underline{c}}^{\bar{c}}(r)\,dr \tag{18}$$

Then, by substituting from equations (17 & 18) in (16) it can be concluded that:

$$T^{-3/2}\sum_{t=1}^{T} y_{t-2} \xrightarrow{d} \lambda \int_0^1 W_{\underline{c}}^{\bar{c}}(r)\,dr \quad , \quad (y_{-1} = 0)$$

Part (3)

From equation (13);

$$T^{-2}\sum_{t=1}^{T} y_{t-2}^2 = T^{-2}\sum_{t=1}^{T} y_{t-1}^2 - 2T^{-2}\sum_{t=1}^{T} y_{t-1}u_{t-1} + T^{-2}\sum_{t=1}^{T} u_{t-1}^2 \tag{19}$$

From equation (11) the order of convergence of $\sum_{t=1}^{T} y_{t-1}^2 = O_P(T^2)$ and the order of convergence of $\sum_{t=1}^{T} u_{t-1}^2 = O_P(T)$ then:

$$\left.\begin{array}{l} 1)\, T^{-2}\sum_{t=1}^{T} y_{t-1}u_{t-1} \xrightarrow{d} 0 \\ 2)\, T^{-2}\sum_{t=1}^{T} u_{t-1}^2 \xrightarrow{d} 0 \end{array}\right\} \tag{20}$$

By using equation (9) then:

$$T^{-2}\sum_{t=1}^{T} y_{t-1}^2 \xrightarrow{d} \lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr \tag{21}$$

Then, by substituting from equations (20 & 21) in (19), it can be concluded that:

$$T^{-2}\sum_{t=1}^{T} y_{t-2}^2 \xrightarrow{d} \lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr \quad , \quad (y_{-1} = 0)$$

Part (4)

From equation (13);

$$T^{-2}\sum_{t=1}^{T} y_{t-1}y_{t-2} = T^{-2}\sum_{t=1}^{T} y_{t-1}^2 - T^{-2}\sum_{t=1}^{T} y_{t-1}u_{t-1} \tag{22}$$

Then, by substituting from equations (20 (1) & 21) in (22) it can be concluded that:

$$T^{-2}\sum_{t=1}^{T} y_{t-1}\, y_{t-2} \xrightarrow{d} \lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr$$

Lemma (2): For model (1) and under the test $H_0: \alpha = 0, \rho_1 = 1, \rho_2 = 0$, then the asymptotic distributions of $T^{1/2}\hat{\alpha}$, $T(\hat{\rho}_1 - 1)$ and $T\hat{\rho}_2$ will be as follows:

$$1) T^{1/2}\hat{\alpha} \xrightarrow{d} \frac{[\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr]\,[\lambda\, W_{\underline{c}}^{\bar{c}}(1)] - [\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\}]}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2}$$

$$2) T(\hat{\rho}_1 - 1) \xrightarrow{d} \frac{[-\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\lambda W_{\underline{c}}^{\bar{c}}(1)] + \frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\}}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2} - z_3$$

$$3) T\hat{\rho}_2 \xrightarrow{d} z_3\,,\ z_3 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T}$$

Proof:

Model (1) can be rewritten in matrix form as follows:

$$Y = X\beta + \mathbf{u} \tag{23}$$

Where:

$$\beta = \begin{bmatrix} \alpha \\ \rho_1 \\ \rho_2 \end{bmatrix},\ X = \begin{bmatrix} 1 & y_0 & y_{-1} \\ 1 & y_1 & y_0 \\ 1 & \vdots & \vdots \\ 1 & y_{T-1} & y_{T-2} \end{bmatrix},\ Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix},\ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}$$

The OLS Estimators of $\hat{\alpha}, \hat{\rho}_1, \hat{\rho}_2$ **are:**

$$\hat{\beta} = (X'X)^{-1} X'Y$$

By using equation (23):

$$\hat{\beta} - \beta = (X'X)^{-1} X'\mathbf{u}$$

Under the null hypothesis that $H_0: \alpha = 0, \rho_1 = 1, \rho_2 = 0$ or $\beta' = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$ then:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\rho}_1 - 1 \\ \hat{\rho}_2 \end{pmatrix} = \begin{pmatrix} T & \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-2} \\ \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-1}^2 & \sum_{t=1}^{T} y_{t-1} y_{t-2} \\ \sum_{t=1}^{T} y_{t-2} & \sum_{t=1}^{T} y_{t-1} y_{t-2} & \sum_{t=1}^{T} y_{t-2}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^{T} u_t \\ \sum_{t=1}^{T} y_{t-1} u_t \\ \sum_{t=1}^{T} y_{t-2} u_t \end{pmatrix} \tag{24}$$

From equation (11) the order of convergence of T, $\sum_{t=1}^{T} u_t$, $\sum_{t=1}^{T} y_{t-1} u_t$, $\sum_{t=1}^{T} y_{t-1}$ ($\sum_{t=2}^{T} y_{t-2}$) and $\sum_{t=1}^{T} y_{t-1}^2$ ($\sum_{t=2}^{T} y_{t-2}^2$) will be $O_p(T)$, $O_p(T^{1/2})$, $O_p(T)$, $O_p(T^{3/2})$ and $O_p(T^2)$ respectively. Also, from equation (12 (1&4))) and by using equation (2) then the order of convergence of $\sum_{t=1}^{T} y_{t-2} u_t$, and $\sum_{t=1}^{T} y_{t-1} y_{t-2}$ will be $O_P(T)$ and $O_P(T^2)$ respectively.

Then, the order of convergence of the elements in equation (24) will be as follows:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\rho}_1 - 1 \\ \hat{\rho}_2 \end{pmatrix} = \begin{pmatrix} O_p(T) & O_p(T^{3/2}) & O_p(T^{3/2}) \\ O_p(T^{3/2}) & O_p(T^2) & O_p(T^2) \\ O_p(T^{3/2}) & O_p(T^2) & O_p(T^2) \end{pmatrix}^{-1} \begin{pmatrix} O_p(T^{1/2}) \\ O_p(T) \\ O_p(T) \end{pmatrix}$$

To obtain the asymptotic distributions of the estimators equation (24) will be multiplied by the following scaling matrix:

$$\psi_T = \begin{pmatrix} T^{1/2} & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{pmatrix}$$

Then equation (24) will be:

$$\psi_T(\hat{\beta} - \beta) = \left\{ \psi_T^{-1} (X'X) \psi_T^{-1} \right\}^{-1} \psi_T^{-1} X' \mathbf{u}$$

$$\begin{pmatrix} T^{1/2} \hat{\alpha} \\ T(\hat{\rho}_1 - 1) \\ T \hat{\rho}_2 \end{pmatrix} = \begin{pmatrix} 1 & T^{-3/2} \sum_{t=1}^{T} y_{t-1} & T^{-3/2} \sum_{t=1}^{T} y_{t-2} \\ T^{-3/2} \sum_{t=1}^{T} y_{t-1} & T^{-2} \sum_{t=1}^{T} y_{t-1}^2 & T^{-2} \sum_{t=1}^{T} y_{t-1} y_{t-2} \\ T^{-3/2} \sum_{t=1}^{T} y_{t-2} & T^{-2} \sum_{t=1}^{T} y_{t-1} y_{t-2} & T^{-2} \sum_{t=1}^{T} y_{t-2}^2 \end{pmatrix}^{-1} \begin{pmatrix} T^{-1/2} \sum_{t=1}^{T} u_t \\ T^{-1} \sum_{t=1}^{T} y_{t-1} u_t \\ T^{-1} \sum_{t=1}^{T} y_{t-2} u_t \end{pmatrix} \tag{25}$$

Form equation (10), $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} u_t \xrightarrow{d} \lambda\, W_{\underline{c}}^{\bar{c}}(1)$, from equation (9), $T^{-1} \sum_{t=1}^{T} y_{t-1} u_t$, $T^{-3/2} \sum_{t=1}^{T} y_{t-1}$ and $T^{-2} \sum_{t=1}^{T} y_{t-1}^2$ convergence in distribution to $\frac{1}{2}\{ \lambda^2 [W(1)_{\underline{c}}^{\bar{c}}]^2 - \gamma_0 \}$, $\lambda \int_0^1 W(r)_{\underline{c}}^{\bar{c}} dr$ and $\lambda^2 \int_0^1 [W(r)_{\underline{c}}^{\bar{c}}]^2 dr$ respectively. Also, from equation (12), $T^{-1} \sum_{t=1}^{T} y_{t-2} u_t$, $T^{-3/2} \sum_{t=1}^{T} y_{t-2}$ and $T^{-2} \sum_{t=1}^{T} y_{t-2}^2$ ($T^{-2} \sum_{t=1}^{T} y_{t-1} y_{t-2}$) convergence in distribution to $\frac{1}{2}\{ \lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0 \} - \gamma_1$, $\lambda \int_0^1 W_{\underline{c}}^{\bar{c}}(r)\, dr$ and $\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr$ respectively.

Then, as $T \to \infty$ and by using the above results equation (25) will be as follows:

$$x_5 = A_5^{-1} h_5 \;,\; x_5 \in R_{(3\times 1)} \text{ (i.e. vector of order } (3\times 1) \text{ of real numbers)} \qquad (26)$$

Where:

$$x_5 = \lim_{T\to\infty}\begin{pmatrix} T^{1/2}\,\hat{\alpha} \\ T(\hat{\rho}_1 - 1) \\ T\,\hat{\rho}_2 \end{pmatrix},\; A_5 = \begin{pmatrix} 1 & \lambda\int_0^1 \bar{W}_{\underline{c}}^{c}(r)\,dr & \lambda\int_0^1 \bar{W}_{\underline{c}}^{c}(r)\,dr \\ \lambda\int_0^1 \bar{W}_{\underline{c}}^{c}(r)\,dr & \lambda^2\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\,dr & \lambda^2\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\,dr \\ \lambda\int_0^1 \bar{W}_{\underline{c}}^{c}(r)\,dr & \lambda^2\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\,dr & \lambda^2\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\,dr \end{pmatrix}$$

$$\text{and } h_5 = \begin{pmatrix} \lambda\,\bar{W}_{\underline{c}}^{c}(1) \\ \frac{1}{2}\{\lambda^2[\bar{W}_{\underline{c}}^{c}(1)]^2 - \gamma_0\} \\ \frac{1}{2}\{\lambda^2[\bar{W}_{\underline{c}}^{c}(1)]^2 - \gamma_0\} - \gamma_1 \end{pmatrix}.$$

Since the value of the determinant of A_5 is equal to zero, a generalized inverse for A_5 is need to be used. There is a generalized inverse G_{51} of A_5 will obtained by using equation (3) as follows:

$$G_{51} = \frac{1}{\lambda^2\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]^2\,dr - \left\{\lambda\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]\,dr\right\}^2}\begin{pmatrix} \lambda^2\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]^2\,dr & -\lambda\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]\,dr & 0 \\ -\lambda\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]\,dr & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{(3\times 3)}$$

Now to obtain the forms of elements of x_5 in equation (26), since:

$$G_{51}\,A_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } G_{51}\,h_5 = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix}$$

Where:

$$\delta_1 = \frac{[\lambda^2\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]^2\,dr]\,[\lambda\,\bar{W}_{\underline{c}}^{c}(1)] - [\lambda\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]\,dr\,][\frac{1}{2}\{\lambda^2[\bar{W}_{\underline{c}}^{c}(1)]^2 - \gamma_0\}]}{\lambda^2\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]^2\,dr - \left\{\lambda\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]\,dr\right\}^2}$$

$$\delta_2 = \frac{[-\lambda\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]\,dr\,][\lambda\,\bar{W}_{\underline{c}}^{c}(1)] + \frac{1}{2}\{\lambda^2[\bar{W}_{\underline{c}}^{c}(1)]^2 - \gamma_0\}}{\lambda^2\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]^2\,dr - \left\{\lambda\int_0^1[\bar{W}_{\underline{c}}^{c}(r)]\,dr\right\}^2}$$

$$\delta_3 = 0$$

Then, by using equation (4) it can be concluded that:

$$x_5 = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} + \left\{ I_3 - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

Where $z's$ **are real numbers, then the asymptotic distributions of**

$T^{1/2}\hat{\alpha}$, $T(\hat{\rho}_1 - 1)$ and $T\hat{\rho}_2$ **will be as follows:**

$$\left.\begin{aligned}
&1)\, T^{1/2}\hat{\alpha} \xrightarrow{d} \\
&\quad \frac{[\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr]\,[\lambda W_{\underline{c}}^{\bar{c}}(1)] - [\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\}]}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2} \\
&2)\, T(\hat{\rho}_1 - 1) \xrightarrow{d} \frac{[-\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\lambda W_{\underline{c}}^{\bar{c}}(1)] + \frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\}}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2} - z_3 \\
&3)\, T\hat{\rho}_2 \xrightarrow{d} z_3 ,\ z_3 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T}
\end{aligned}\right\} \quad (27)$$

Corollary (1): If there is another generalized inverse G_{52} of A_5 **that** can be obtained by using equation (3), it will be as follows:

$$G_{52} = \frac{1}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2} \begin{pmatrix} \lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr & 0 & -\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr \\ 0 & 0 & 0 \\ -\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr & 0 & 1 \end{pmatrix}_{(3\times 3)}$$

Then, **the asymptotic distributions of** $T^{1/2}\hat{\alpha}$, $T(\hat{\rho}_1 - 1)$ and$T\hat{\rho}_2$ **will be as follows:**

$$\left.\begin{aligned}
&1)\, T^{1/2}\hat{\alpha} \xrightarrow{d} \\
&\quad \frac{[\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr]\,[\lambda W_{\underline{c}}^{\bar{c}}(1)] - [\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\} - \gamma_1]}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2} \\
&2)\, T(\hat{\rho}_1 - 1) \xrightarrow{d} z_2 \\
&3)\, T\hat{\rho}_2 \xrightarrow{d} \frac{[-\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\lambda W_{\underline{c}}^{\bar{c}}(1)] + \frac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\} - \gamma_1}{\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2 dr - \left\{\lambda \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2} - z_2 \\
&\qquad ,z_2 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T}
\end{aligned}\right\} \quad (28)$$

2.2 Asymptotic distributions of the $t-type$ statistics under different tests of hypothesis

In addition to the previous tests in (2.1), the tests that based on $t-type$ statistics for the estimators $\hat{\alpha}$, $\hat{\rho}_1$ and $\hat{\rho}_2$ under the test $H_0:\alpha=0, \rho_1=1, \rho_2=0$, (i.e. $y_t = y_{t-1}+u_t$) against $H_a:\alpha\neq 0$, $|\rho_1|<1$, $|\rho_2|<1$, (i.e. $y_t=\alpha+\rho_1 y_{t-1}+\rho_2 y_{t-2}+u_t$) **will be** derived **as follows:**

Lemma (3): If the variance-covariance matrix of the estimators of model (1) under the **null hypothesis** $H_0:\alpha=0, \rho_1=1$, $\rho_2=0$ **that** can be written in matrix form as follows:

$$Var(\hat{\beta}) = S_T'^2 (X'X)^{-1} \tag{29}$$

Such that,

$$\left.\begin{aligned}
&1) Var(\hat{\beta}) = \begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\rho}_1,\hat{\alpha}) & Cov(\hat{\rho}_2,\hat{\alpha}) \\ Cov(\hat{\rho}_1,\hat{\alpha}) & Var(\hat{\rho}_1) & Cov(\hat{\rho}_1,\hat{\rho}_2) \\ Cov(\hat{\rho}_2,\hat{\alpha}) & Cov(\hat{\rho}_1,\hat{\rho}_2) & Var(\hat{\rho}_2) \end{pmatrix} \\
&2) (X'X)^{-1} = \begin{pmatrix} T & \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-2} \\ \sum_{t=1}^{T} y_{t-1} & \sum_{t=1}^{T} y_{t-1}^2 & \sum_{t=1}^{T} y_{t-1}y_{t-2} \\ \sum_{t=1}^{T} y_{t-2} & \sum_{t=1}^{T} y_{t-1}y_{t-2} & \sum_{t=1}^{T} y_{t-2}^2 \end{pmatrix}^{-1} \\
&3) S_T'^2 = \sum_{t=1}^{T} \hat{u}_t^2 / (T-3)
\end{aligned}\right\} \tag{30}$$

Then, the asymptotic distributions f**or** $t_{\hat{\alpha}}$, $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ **will be as follows:**

$$1)\, t_{\hat{\alpha}} = [T^{1/2}(\hat{\alpha})][T\, Var(\hat{\alpha})]^{-1/2} \xrightarrow{d} \delta_1\, \mathrm{d}_1^{-1/2}$$

$$2)\, t_{\hat{\rho}_1} = [T(\hat{\rho}_1-1)][T^2\, Var(\hat{\rho}_1)]^{-1/2} \xrightarrow{d} (\delta_2 - z_3)\, \mathrm{d}_2^{-1/2}$$

$$3)\, t_{\hat{\rho}_2} = [T\hat{\rho}_2][T^2\, Var(\hat{\rho}_2)]^{-1/2} \xrightarrow{d} z_3\, \mathrm{d}_3^{-1/2}, z_3 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T}, z_{32} \in \underline{c}\sqrt{T}, z_{33} \in \bar{c}\sqrt{T}$$

Where δ_1 , δ_2 are defined as in lemma (2), $\mathrm{d}_1 = \dfrac{\gamma_0 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left\{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2}$

$$,\mathrm{d}_2 = \frac{\gamma_0 - \gamma_0^2\, z_{32}}{\lambda^2\left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left\{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2\right)}\ ,\ \mathrm{d}_3 = \frac{\gamma_0^2\, z_{33}}{\lambda^2\left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left\{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2\right)}$$

Proof:

By multiplying equation (29) by ψ_T that defined as in lemma (2),then;

$$\psi_T Var(\hat{\beta})\psi_T = S_T'^2 (\psi_T^{-1} X' X \psi_T^{-1})^{-1} \tag{31}$$

So, by substituting from equation (30 (1,2)) in (31) then the variance- covariance matrix will be:

$$\begin{pmatrix} T\,Var(\hat{\alpha}) & T^{3/2} Cov(\hat{\rho}_1,\hat{\alpha}) & T^{3/2} Cov(\hat{\rho}_2,\hat{\alpha}) \\ T^{3/2} Cov(\hat{\rho}_1,\hat{\alpha}) & T^2 Var(\hat{\rho}_1) & T^2 Cov(\hat{\rho}_1,\hat{\rho}_2) \\ T^{3/2} Cov(\hat{\rho}_2,\hat{\alpha}) & T^2 Cov(\hat{\rho}_1,\hat{\rho}_2) & T^2 Var(\hat{\rho}_2) \end{pmatrix} = S_T'^2 B_3 \tag{32}$$

Where:

$$B_3 = \begin{pmatrix} 1 & T^{-3/2}\sum_{t=1}^T y_{t-1} & T^{-3/2}\sum_{t=1}^T y_{t-2} \\ T^{-3/2}\sum_{t=1}^T y_{t-1} & T^{-2}\sum_{t=1}^T y_{t-1}^2 & T^{-2}\sum_{t=1}^T y_{t-1}y_{t-2} \\ T^{-3/2}\sum_{t=1}^T y_{t-2} & T^{-2}\sum_{t=1}^T y_{t-1}y_{t-2} & T^{-2}\sum_{t=1}^T y_{t-2}^2 \end{pmatrix}^{-1}$$

As $T \to \infty$ and from the weak law of large number, Bell [9], and from equation (13 (1)) then the convergence in probability of $S_T'^2$ will be as follows:

$$S_T'^2 = \sum_{t=1}^T \hat{u}_t^2 / (T-3) \xrightarrow{p} \gamma_0 \tag{33}$$

From equations (9 (3,4)), $T^{-3/2}\sum_{t=1}^T y_{t-1}$ and $T^{-2}\sum_{t=1}^T y_{t-1}^2$ convergence in distribution to $\lambda \int_0^1 W(r)_{\underline{c}}^{\bar{c}}\, dr$ and $\lambda^2 \int_0^1 [W(r)_{\underline{c}}^{\bar{c}}]^2\, dr$ respectively. Also, from equations (12 (2,3,4)), $T^{-3/2}\sum_{t=1}^T y_{t-2}$ and $T^{-2}\sum_{t=1}^T y_{t-2}^2$ ($T^{-2}\sum_{t=1}^T y_{t-1}y_{t-2}$) convergence in distribution to $\lambda \int_0^1 W_{\underline{c}}^{\bar{c}}(r)\, dr$ and $\lambda^2 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr$ respectively.

Then, as $T \to \infty$ **,** by using the above results equation (32) will be:

$$x_6 = A_6^{-1} h_6 \,,\; x_6 \in R_{(3\times 3)} \tag{34}$$

Where:

$$x_6 = \lim_{T\to\infty} \begin{pmatrix} T\,Var(\hat{\alpha}) & T^{3/2} Cov(\hat{\rho}_1,\hat{\alpha}) & T^{3/2} Cov(\hat{\rho}_2,\hat{\alpha}) \\ T^{3/2} Cov(\hat{\rho}_1,\hat{\alpha}) & T^2 Var(\hat{\rho}_1) & T^2 Cov(\hat{\rho}_1,\hat{\rho}_2) \\ T^{3/2} Cov(\hat{\rho}_2,\hat{\alpha}) & T^2 Cov(\hat{\rho}_1,\hat{\rho}_2) & T^2 Var(\hat{\rho}_2) \end{pmatrix},\; h_6 = I_3 ,$$

$$A_6 = \begin{pmatrix} \frac{1}{\gamma_0} & \frac{\lambda}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]\, dr & \frac{\lambda}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]\, dr \\ \frac{\lambda}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]\, dr & \frac{\lambda^2}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\, dr & \frac{\lambda^2}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\, dr \\ \frac{\lambda}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]\, dr & \frac{\lambda^2}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\, dr & \frac{\lambda^2}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\, dr \end{pmatrix}$$

and A_6 is the asymptotic distribution of the matrix $S'^2_T\, B_3$.

Since, $\left|A_6\right| = 0$ a generalize0d inverse G_{61} of A_6 will obtained by using equation (3) and it will be as:

$$G_{61} = \frac{1}{\frac{\lambda^2}{\gamma_0^2}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\, dr - \left\{\frac{\lambda}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]\, dr\right\}^2} \begin{pmatrix} \frac{\lambda^2}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\, dr & -\frac{\lambda}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]\, dr & 0 \\ -\frac{\lambda}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]\, dr & \frac{1}{\gamma_0} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{(3\times 3)}$$

Now to obtain the forms of elements of x_6 in equation (37), equation (4) will be used, the forms of the asymptotic distributions of $T\,Var(\hat{\alpha}),\ T^2\,Var(\hat{\rho}_1)$ and $T^2\,Var(\hat{\rho}_2)$, and the asymptotic distributions for $t_{\hat{\alpha}}$, $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ will be derived as follows:

Since:

$$G_{61}\, A_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$G_{61}\, h_6 = \frac{1}{\frac{\lambda^2}{\gamma_0^2}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\, dr - \left\{\frac{\lambda}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]\, dr\right\}^2} \begin{pmatrix} \frac{\lambda^2}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]^2\, dr & -\frac{\lambda}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]\, dr & 0 \\ -\frac{\lambda}{\gamma_0}\int_0^1 [\bar{W}_{\underline{c}}^{c}(r)]\, dr & \frac{1}{\gamma_0} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, by using equation (4) it can be concluded that:

$$x_6 = \frac{1}{\frac{\lambda^2}{\gamma_0^2}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr - \left\{\frac{\lambda}{\gamma_0}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr\right\}^2}\begin{pmatrix} \frac{\lambda^2}{\gamma_0}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr & -\frac{\lambda}{\gamma_0}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr & 0 \\ -\frac{\lambda}{\gamma_0}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr & \frac{1}{\gamma_0} & 0 \\ 0 & 0 & 0 \end{pmatrix} +$$

$$\left\{ I_3 - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix}$$

Where $z's$ are real numbers, then the asymptotic distributions of $T\,Var(\hat{\alpha}), T^2 Var(\hat{\rho}_1)$

and $T^2 Var(\hat{\rho}_2)$ will be as follows:

$$\left.\begin{aligned}
&1) T\,Var(\hat{\alpha}) \xrightarrow{d} \frac{\gamma_0 \int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr - \left\{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr\right\}^2} > 0 \\
&2) T^2 Var(\hat{\rho}_1) \xrightarrow{d} \frac{\gamma_0 - \gamma_0^2 z_{32}}{\lambda^2\left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr - \left\{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr\right\}^2\right)} > 0 \\
&3) T^2 Var(\hat{\rho}_2) \xrightarrow{d} \frac{\gamma_0^2\, z_{33}}{\lambda^2\left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\,dr - \left\{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\,dr\right\}^2\right)} > 0
\end{aligned}\right\} \quad (35)$$

To achieve the variances in equation (35) to be positive, $z_{32} \le 0$ and it is assumed to be $z_{32} \in \underline{c}\sqrt{T}$ and $z_{33} > 0$ and it is assumed to be $z_{33} \in \bar{c}\sqrt{T}$.

The $t-type$ statistics for the estimators $\hat{\alpha}$, $\hat{\rho}_1$ and $\hat{\rho}_2$ will be obtained as:

$$\left.\begin{aligned}
&1) t_{\hat{\alpha}} = [T^{1/2}(\hat{\alpha})][T\,Var(\hat{\alpha})]^{-1/2} \\
&2) t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2 Var(\hat{\rho}_1)]^{-1/2} \\
&3) t_{\hat{\rho}_2} = [T\,\hat{\rho}_2][T^2 Var(\hat{\rho}_2)]^{-1/2}
\end{aligned}\right\} \quad (36)$$

Then, by substituting from equation (27) that contains the asymptotic distributions of OLS estimators $T^{1/2}\hat{\alpha}, T(\hat{\rho}_1 - 1)$ and $T\hat{\rho}_2$, and (35) in (36) then the asymptotic distributions f**or** $t_{\hat{\alpha}}$, $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ respectively **will be**:

$$\left.\begin{aligned}
&1)\, t_{\hat{\alpha}} = [T^{1/2}(\hat{\alpha})][T\,Var(\hat{\alpha})]^{-1/2} \xrightarrow{d} \delta_1\, \mathrm{d}_1^{-1/2}\\
&2)\, t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2 Var(\hat{\rho}_1)]^{-1/2} \xrightarrow{d} (\delta_2 - z_3)\, \mathrm{d}_2^{-1/2}\\
&3)\, t_{\hat{\rho}_2} = [T\,\hat{\rho}_2][T^2 Var(\hat{\rho}_2)]^{-1/2} \xrightarrow{d} z_3\, \mathrm{d}_3^{-1/2},\\
&\qquad z_3 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T},\, z_{32} \in \underline{c}\sqrt{T},\, z_{33} \in \bar{c}\sqrt{T}
\end{aligned}\right\} \quad (37)$$

Corollary (2): If there is another generalized inverse G_{62} **of** A_6 that **can be obtained by using equation (3), it will be as follows:**

$$G_{62} = \frac{1}{\frac{\lambda^2}{\gamma_0^2}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left\{\frac{\lambda}{\gamma_0}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2} \begin{pmatrix} \frac{\lambda^2}{\gamma_0}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr & 0 & -\frac{\lambda}{\gamma_0}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr \\ 0 & 0 & 0 \\ -\frac{\lambda}{\gamma_0}\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr & 0 & \frac{1}{\gamma_0} \end{pmatrix}_{(3\times 3)}$$

Then, the asymptotic distributions f**or** $t_{\hat{\alpha}}$, $t_{\hat{\rho}_1}$ and $t_{\hat{\rho}_2}$ **will be**:

$$\left.\begin{aligned}
&1)\, t_{\hat{\alpha}} = [T^{1/2}(\hat{\alpha})][T\,Var(\hat{\alpha})]^{-1/2} \xrightarrow{d} \delta_4\, \mathrm{d}_4^{-1/2}\\
&2)\, t_{\hat{\rho}_1} = [T(\hat{\rho}_1 - 1)][T^2 Var(\hat{\rho}_1)]^{-1/2} \xrightarrow{d} z_2\, \mathrm{d}_5^{-1/2}\\
&3)\, t_{\hat{\rho}_2} = [T\,\hat{\rho}_2][T^2 Var(\hat{\rho}_2)]^{-1/2} \xrightarrow{d} (\delta_5 - z_2)\, \mathrm{d}_6^{-1/2},\\
&\qquad z_2 \in \underline{c}\sqrt{T} \text{ or } \bar{c}\sqrt{T},\, z_{23} \in \underline{c}\sqrt{T},\, z_{22} \in \bar{c}\sqrt{T}
\end{aligned}\right\} \quad (38)$$

Where:

$$\delta_4 = \frac{[\lambda^2\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr]\,[\lambda\, W_{\underline{c}}^{\bar{c}}(1)] - [\lambda\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\tfrac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\} - \gamma_1]}{\lambda^2\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left\{\lambda\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2}$$

$$\delta_5 = \frac{[-\lambda\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr][\lambda W_{\underline{c}}^{\bar{c}}(1)] + \tfrac{1}{2}\{\lambda^2 [W_{\underline{c}}^{\bar{c}}(1)]^2 - \gamma_0\} - \gamma_1}{\lambda^2\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left\{\lambda\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2}$$

$$\mathrm{d}_4 = \frac{\gamma_0\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr}{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left\{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2},\quad \mathrm{d}_5 = \frac{\gamma_0^2\, z_{22}}{\lambda^2\left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left\{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2\right)},$$

$$\mathrm{d}_6 = \frac{\gamma_0 - \gamma_0^2\, z_{23}}{\lambda^2\left(\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]^2\, dr - \left\{\int_0^1 [W_{\underline{c}}^{\bar{c}}(r)]\, dr\right\}^2\right)}$$

3 Simulation Study

A simulation study is used to obtain MSE and Thiel's U under the null hypothesis $H_0: y_t = y_{t-1} + u_t$ and under the alternative hypothesis H_a with constant term will be obtained in case of five samples size T = 30, 50, 100, 200 and 500 for five boundaries value $\bar{c} = -\underline{c} = 0.3, 0.5, 0.7, 0.9 \text{ and } 1.1$ in case of ten values for the coefficient of dependent errors $\phi_1 = \pm 0.5, \pm 0.4, \pm 0.3, \pm 0.2 \text{ and} \pm 0.1$ by 5000 replications as follows:

OLS estimators of bounded AR (2) with constant model in case of dependent errors which obtained in lemma (2) that used the generalized inverse G_{51} and in corollary (1) that used the generalized inverse G_{52} are used to obtain MSE and Thiel's U and the results can be summarized and discussed for the next five cases:

Case (1): T = 30

Table 1. Alternative hypothesis H_a for all values of of $\bar{c} = -\underline{c}$ (T= 30)

$\bar{c} = -\underline{c}$		ϕ_1 / Criteria	0.5	0.4	0.3	0.2	0.1	-0.1	-0.2	-0.3	-0.4	-0.5
G_{51}	0.3	MSE	H_0					H_a				
		Thiel's U	H_a									
	0.5	MSE	H_0					H_a				
		Thiel's U	H_a									
	0.7	MSE										
		Thiel's U										
	0.9	MSE										
		Thiel's U										
	1.1	MSE										
		Thiel's U										
G_{52}	0.3	MSE	H_0									
		Thiel's U										
	0.5	MSE	H_a					H_0				
		Thiel's U										
	0.7	MSE	H_a								H_0	
		Thiel's U	H_a						H_0			
	0.9	MSE	H_a									
		Thiel's U	H_a								H_0	
	1.1	MSE	H_a									
		Thiel's U	H_a									H_0

It can be notice from Table (1) that G_{51} approve the alternative hypothesis H_a for all values of $\bar{c} = -\underline{c}$ except for values of MSE, $\bar{c} = -\underline{c} = 0.3 \text{ and } 0.5$ in case of positive values of ϕ_1 and G_{52} approve the alternative hypothesis H_a for most values of $\bar{c} = -\underline{c}$ except for the values of MSE, Thiel's U and $\bar{c} = -\underline{c} = 0.3$ for all values of ϕ_1 and for the values of MSE, Thiel's U and $\bar{c} = -\underline{c} = 0.5$ in case of negative values of ϕ_1 .

Case (2): T = 50

Table 2. Alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ (T= 50)

$\bar{c}=-\underline{c}$		ϕ_1 / Criteria	0.5	0.4	0.3	0.2	0.1	-0.1	-0.2	-0.3	-0.4	-0.5
G_{51}	0.3	MSE	H_0					H_a				
		Thiel's U	H_a									
	0.5	MSE										
		Thiel's U										
	0.7	MSE										
		Thiel's U										
	0.9	MSE										
		Thiel's U										
	1.1	MSE										
		Thiel's U										
G_{52}	0.3	MSE	H_a					H_0				
		Thiel's U	H_a			H_0						
	0.5	MSE	H_a							H_0		
		Thiel's U	H_a					H_0				
	0.7	MSE	H_a									H_0
		Thiel's U	H_a						H_0			
	0.9	MSE	H_a									
		Thiel's U	H_a								H_0	
	1.1	MSE	H_a									
		Thiel's U										

It can be notice from Table (2) that G_{51} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for values of MSE, $\bar{c}=-\underline{c}=0.3$ in case of positive values of ϕ_1 and G_{52} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of MSE, Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of negative values of ϕ_1 and for the values of Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of negative values of ϕ_1 .

Case (3): T = 100

Table 3. Alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ (T= 100)

$\bar{c}=-\underline{c}$		ϕ_1 / Criteria	0.5	0.4	0.3	0.2	0.1	-0.1	-0.2	-0.3	-0.4	-0.5
G_{51}	0.3	MSE	H_0			H_a						
		Thiel's U	H_a									
	0.5	MSE										
		Thiel's U										
	0.7	MSE										
		Thiel's U										
	0.9	MSE										
		Thiel's U										
	1.1	MSE										
		Thiel's U										

$\bar{c}=-\underline{c}$		ϕ_1 / **Criteria**	**0.5**	**0.4**	**0.3**	**0.2**	**0.1**	**-0.1**	**-0.2**	**-0.3**	**-0.4**	**-0.5**
G_{52}	0.3	MSE	H_a					H_0				
		Thiel's U	H_a			H_0						
	0.5	MSE	H_a								H_0	
		Thiel's U	H_a					H_0				
	0.7	MSE	H_a									
		Thiel's U	H_a							H_0		
	0.9	MSE	H_a									
		Thiel's U	H_a									H_0
	1.1	MSE	H_a									
		Thiel's U										

It can be notice from Table 3 that G_{51} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ except for values of MSE, $\bar{c}=-\underline{c}=0.3$ in case of $\phi_1=0.5, 0.4$ and 0.3 and G_{52} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of MSE, Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of negative values of ϕ_1 and for the values of Thiel's U and $\bar{c}=-\underline{c}=0.5$ in case of negative values of ϕ_1 .

Case (4): T = 200

Table 4. Alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ (T= 200)

$\bar{c}=-\underline{c}$		ϕ_1 / **Criteria**	**0.5**	**0.4**	**0.3**	**0.2**	**0.1**	**-0.1**	**-0.2**	**-0.3**	**-0.4**	**-0.5**
G_{51}	0.3	MSE	H_a									
		Thiel's U										
	0.5	MSE										
		Thiel's U										
	0.7	MSE										
		Thiel's U										
	0.9	MSE										
		Thiel's U										
	1.1	MSE										
		Thiel's U										
G_{52}	0.3	MSE	H_a							H_0		
		Thiel's U	H_a				H_0					
	0.5	MSE	H_a									H_0
		Thiel's U	H_a					H_0				
	0.7	MSE	H_a									
		Thiel's U	H_a								H_0	
	0.9	MSE	H_a									
		Thiel's U										
	1.1	MSE										
		Thiel's U										

It can be notice from Table (4) that G_{51} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ and G_{52} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of Thiel's U and $\bar{c}=-\underline{c}=0.3$ and 0.5 in case of negative values of ϕ_1 .

Case (5): T = 500

It can be notice from Table (5) that G_{51} approve the alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ and G_{52} approve the alternative hypothesis H_a for most values of $\bar{c}=-\underline{c}$ except for the values of Thiel's U and $\bar{c}=-\underline{c}=0.3$ in case of negative values of ϕ_1 .

Table 5. Alternative hypothesis H_a for all values of $\bar{c}=-\underline{c}$ (T= 500)

$\bar{c}=-\underline{c}$		ϕ_1 / Criteria	**0.5**	**0.4**	**0.3**	**0.2**	**0.1**	**-0.1**	**-0.2**	**-0.3**	**-0.4**	**-0.5**
G_{51}	0.3	MSE	H_a									
		Thiel's U										
	0.5	MSE										
		Thiel's U										
	0.7	MSE										
		Thiel's U										
	0.9	MSE										
		Thiel's U										
	1.1	MSE										
		Thiel's U										
G_{52}	0.3	MSE	H_a							H_0		
		Thiel's U	H_a					H_0				
	0.5	MSE	H_a									H_0
		Thiel's U	H_a							H_0		
	0.7	MSE	H_a									
		Thiel's U	H_a									H_0
	0.9	MSE	H_a									
		Thiel's U										
	1.1	MSE										
		Thiel's U										

4 Conclusion

The asymptotic distributions of OLS estimators of bounded AR (2) model with constant term in case of dependent errors under different tests of hypothesis have been derived. Also, the asymptotic distributions of the $t-type$ statistics of OLS estimators have been derived.

The measurement of MSE approve H_a more than the measurement of Thiel's U. Also, the positive values of ϕ_1 approve H_a more than the negative values of ϕ_1.

The generalized inverse G_{51} approve H_a more than the generalized inverse G_{52} in all cases of sample size T, $\bar{c} = -\underline{c}$ and ϕ_1. Also, for each sample size T, $\bar{c} = -\underline{c}$ and for generalized inverses G_{51} and G_{52} the values of MSE are decreasing for decreasing of positive values of ϕ_1 and increasing for decreasing of negative values of ϕ_1, while the values of Thiel's U are increasing for both decreasing of positive values of ϕ_1 and decreasing of negative values of ϕ_1 under both the null and alternative hypotheses.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Dickey DA, Fuller WA. Distribution of the estimators for autoregressive time series with a unit root. JASA. 1979;74(366):427-431.

[2] Dickey DA, Fuller WA. Likelihood ratio statistics for autoregressive time series with a unit. Econometrica. 1981;49(4):1057-1072.

[3] Cavaliere G. A rescaled range statistics approach to unit root tests. Econometric Society World Congress 2000 Contributed Papers 0318; 2000. Available:http://fmwww.bc.edu/RePEc/es2000/0318.pdf.

[4] Cavaliere G. Limited time series with a unit root. Econometric Theory. 2005;21(5):907-945.

[5] Cavaliere G, Xu F. Testing for unit roots in bounded time series. University of Bologna, European University Institute Christian-Albrechts-University of Kiel; 2011. Available:http://www.econ.queensu.ca/files/event/Cavaliere_Xu.pdf.

[6] Schatzman M. Numerical analysis: A mathematical introduction. Clarendon Press, Oxford; 2002.

[7] Sawyer S. Generalized inverses: How to invert a non-invertible. Matrix; 2008. Available:https://www.math.wustl.edu/~sawyer/handouts/GenrlInv.pdf.

[8] Amer GA. Econometrics and time series analysis (Theroy, Methods, Applications) Cairo University; 2015.

[9] Bell J. The weak and strong laws of large numbers. University of Toronto; 2015. Available:https://pdfs.semanticscholar.org/4786/984d97527d81b17ba34bbfdbbb46f1e69f48.pdf

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