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Article

On the Noteworthy Properties of Tangentials in Cubic Structures

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Abstract: The cubic structure, a captivating geometric structure, finds applications across various areas of geometry through different models. In this paper, we explore the significant characteristics of tangentials in cubic structures of ranks 0, 1, and 2. Specifically, in the cubic structure of rank 2, we derive the Hessian configuration $(12_3, 16_4)$ of points and lines. Finally, we introduce and investigate the de Vries configuration of points and lines in a cubic structure.

Keywords: cubic structure; rank of cubic structure; tangential; corresponding points; Hesse configuration; de Vries configuration

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1. Introduction

The close connection between the cubic structure and algebraic structures on cubic curves was studied in [1–3], and this correlation was further examined in [4]. The cubic structure was defined in [4]. Let Q be a nonempty set whose elements are called *points*, and let $[\ ] \subseteq Q^3$ be a ternary relation on Q. Such a relation and the ordered pair $(Q, [\ ])$ will be called a *cubic relation* and a *cubic structure*, respectively, if the following properties are satisfied:

C1. For any two points $a, b \in Q$, there is a unique point $c \in Q$ such that $[a, b, c]$ (i.e., $(a, b, c) \in [\ ]$).
C2. The relation $[\ ]$ is totally symmetric (i.e., $[a, b, c]$ implies $[a, c, b]$, $[b, a, c]$, $[b, c, a]$, $[c, a, b]$, and $[c, b, a]$).
C3. $[a, b, c]$, $[d, e, f]$, $[g, h, i]$, $[a, d, g]$, and $[b, e, h]$ imply $[c, f, i]$, which can be clearly written in the form of the following table:

$$\begin{array}{cc|c|} a & b & c \\ d & e & f \\ g & h & i \end{array}.$$

In [4], numerous examples of cubic structures are presented, with one notable example in which Q is the set of all non-singular points of a cubic curve in the plane. In this context, the notation $[a, b, c]$ signifies that points a, b, and $c \in Q$ are collinear. Therefore, in a general cubic structure $(Q, [\ ])$, if $[a, b, c]$ holds true, then we will also say that points a, b, and c form a *line*. If this statement is not valid, then we say that (a, b, c) is a *triangle*.

The concept of tangentials of points was introduced in [5]. The point a' is said to be the *tangential* of the point a if $[a, a, a']$ holds true. If a' is the tangential of the point a, then we will also say that the point a is an antecedent of the point a'. It is obvious that every point has one and only one tangential a'. The tangential a'' of the tangential a' of a point a is called its *second tangential*. We will always denote the tangential and the second tangential of any point x as x' and x''. The validity of $[a, b, c]$ implies the validity of $[a', b', c']$. Two distinct points having the same tangential are called *corresponding points*. All points such that any two of them are corresponding points (i.e., have the same tangential) are said to be *associated*. The maximal number of associated points always equals 2^m for some $m \in \mathbb{N} \cup \{0\}$, and the number m is called the *rank* of the observed cubic structure. In the case of a cubic structure where collinear triples of non-singular points are observed on a cubic curve in the complex plane, ranks 0, 1, or 2 appear, depending on whether the cubic has a spike, an ordinary double point, or is without singular points. We will mention here some of the results from [5] in the form of several lemmas.

Lemma 1. *Let a_1 and a_2 be corresponding points with the common tangential a', let o be any point, and let b_1 and b_2 be points such that $[o, a_1, b_1]$ and $[o, a_2, b_2]$. Then, b_1 and b_2 are corresponding points with the common tangential b' such that $[o', a', b']$. In addition, there is a point c such that $[a_1, b_2, c]$ and $[a_2, b_1, c]$, and o and c are corresponding points.*

Lemma 2. *If $[a, b, c]$; $[a, e, f]$; $[b, f, d]$; and $[c, d, e]$, then a, d; b, e; and c, f are pairs of corresponding points, and the tangentials a', b', and c' satisfy $[a', b', c']$.*

Lemma 3. *If a_1, a_2, a_3, and a_4 are associated points with the common tangential a', then there exist points p, q, and r such that $[a_1, a_2, p]$, $[a_3, a_4, p]$, $[a_1, a_3, q]$, $[a_2, a_4, q]$, $[a_1, a_4, r]$, and $[a_2, a_3, r]$, and points a', p, q, and r are associated.*

Lemma 4. *Suppose $[a', b', c']$ holds, where a', b', and c' are mutually different points. All different antecedents of points a', b', and c' in a cubic structure of rank 2 can be denoted by a_1, a_2, a_3, and a_4; b_1, b_2, b_3, and b_4; and c_1, c_2, c_3, and c_4 such that the following hold:*

$$\begin{array}{llll} [a_1, b_1, c_1], & [a_1, b_2, c_2], & [a_1, b_3, c_3], & [a_1, b_4, c_4], \\ [a_2, b_1, c_2], & [a_2, b_2, c_1], & [a_2, b_3, c_4], & [a_2, b_4, c_3], \\ [a_3, b_1, c_3], & [a_3, b_2, c_4], & [a_3, b_3, c_1], & [a_3, b_4, c_2], \\ [a_4, b_1, c_4], & [a_4, b_2, c_3], & [a_4, b_3, c_2], & [a_4, b_4, c_1]. \end{array}$$

Points a, b, c, d, e, and f are said to form a *quadrilateral* $\{a, d; b, e; c, f\}$ if there exist lines $[a, b, c]$, $[a, e, f]$, $[d, b, f]$, and $[d, e, c]$, and we say that the points from each pair of points a, d; b, e; and c, f are *opposite*. Lemma 2 actually asserts that pairs of opposite vertices of a quadrilateral are corresponding.

We will say that a, d; b, e; and c, f are pairs of opposite vertices of a *complete quadrilateral* $(a, d; b, e; c, f)$ if there exist lines $[a, b, c]$, $[a, e, f]$, $[d, b, f]$, and $[d, e, c]$. According to [5] (Theorem 3.5), pairs of opposite vertices have common tangentials which are collinear points.

The motivation for this paper is drawn from classical books [6–9] which extensively covered the properties of cubics. Additionally, a wealth of various research papers on this subject exist, although we will not provide a detailed list.

2. Some Properties of Tangentials in a General Cubic Structure

The following theorem allows the construction of the tangential of a point when the tangential of another given point is known:

Theorem 1. *Let a point a and its tangential a' be given. For each point p, let q, r, s, and t be points such that $[a, p, q]$, $[a', p, r]$, $[q, r, s]$, and $[a, s, t]$ hold. Then, point t is the tangential of point p.*

Proof. This statement follows from the table

$$\begin{array}{cc|c|} \cline{3-3} a' & r & p \\ a & q & p \\ a & s & t \\ \cline{3-3} \end{array}. \quad \square$$

Theorem 2. *Let q and r be fixed points, and let a_1, a_2, b_1, and b_2 be points such that $[r, a_1, a_2]$, $[q, a_1, b_1]$, and $[q, a_2, b_2]$. Then, point s such that $[s, b_1, b_2]$ is also a fixed point.*

Proof. Let q' be the tangential of q. From the table

$$\begin{array}{cc|c|}\cline{3-3} q & q & q' \\ a_1 & a_2 & r \\ b_1 & b_2 & s \\ \cline{3-3}\end{array}$$

we obtain $[q', r, s]$. Let c_1, c_2, d_1, and d_2 be points such that $[r, c_1, c_2]$, $[q, c_1, d_1]$, and $[q, c_2, d_2]$. Then, the statement $[s, d_1, d_2]$, which should be proven, follows from the table

$$\begin{array}{cc|c|}\cline{3-3} q & c_1 & d_1 \\ q & c_2 & d_2 \\ q' & r & s \\ \cline{3-3}\end{array}. \quad \square$$

Theorem 3. *If p, q, r, and s are points such that $[a, p, q]$, and $[a, r, s]$ hold, and u and v are points such that $[p, r, u]$ and $[q, s, v]$ hold, then for the tangential a' of point a, the statement $[u, v, a']$ holds.*

Proof. This statement follows from the table

$$\begin{array}{cc|c|}\cline{3-3} p & r & u \\ q & s & v \\ a & a & a' \\ \cline{3-3}\end{array}. \quad \square$$

Theorem 4. *Let a_1, b_1, c_1, and d_1 be points such that there is a point e_1 which satisfies $[a_1, b_1, e_1]$ and $[c_1, d_1, e_1]$, and let t be any point. If points a_2, b_2, c_2, and d_2 satisfy $[t, a_1, a_2]$, $[t, b_1, b_2]$, $[t, c_1, c_2]$, and $[t, d_1, d_2]$, then there is a point e_2 satisfying $[a_2, b_2, e_2]$ and $[c_2, d_2, e_2]$.*

Proof. If t' is the tangential of point t, and e_2 is the point such that $[t', e_1, e_2]$, then from the tables

$$\begin{array}{cc|c|}\cline{3-3} t & a_1 & a_2 \\ t & b_1 & b_2 \\ t' & e_1 & e_2 \\ \cline{3-3}\end{array} \qquad \begin{array}{cc|c|}\cline{3-3} t & c_1 & c_2 \\ t & d_1 & d_2 \\ t' & e_1 & e_2 \\ \cline{3-3}\end{array}$$

we obtain $[a_2, b_2, e_2]$ and $[c_2, d_2, e_2]$. $\square$

Theorem 5. *Let points a_1 and a_2 have the common tangential a', and let b be any point. If c and d are points such that $[a_1, b, c]$ and $[a_2, b, d]$ hold, then there is a point e satisfying $[a_1, d, e]$ and $[a_2, c, e]$.*

Proof. Let point e be such that $[a_1, d, e]$. Then, $[a_2, c, e]$ follows from the table

$$\begin{array}{cc|c|}\cline{3-3} a' & a_2 & a_2 \\ a_1 & b & c \\ a_1 & d & e \\ \cline{3-3}\end{array}. \quad \square$$

Theorem 6. *In a cubic structure, there are as many triangles with vertices a, b, and c whose sides pass through the three given points d, e, and f of this cubic structure as there are antecedents of each point (i.e., in a structure of rank m, there are 2^m such triangles).*

Proof. Let g be the point such that $[e, f, g]$, and let h be the point satisfying $[d, g, h]$. If a is some antecedent point of the point h (i.e., if h is the tangential a' of the point a), then a is one of the required points. Indeed, if b and c are points such that $[a,f,b]$ and $[a,e,c]$, then we obtain $[b,c,d]$ from the table

$$\begin{array}{cc|c|} \cline{3-3} a & f & b \\ a & e & c \\ h & g & d \\ \cline{3-3} \end{array}.$$

The number of solutions is equal to the number of antecedents of point h. □

Theorem 7. *If $[a_1,a_2,a_3]$, $[b_1,b_2,b_3]$, and $[a_i,b_i,c_i]$, where $i=1,2,3$, then the table*

$$\begin{array}{cc|c|} \cline{3-3} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ \cline{3-3} \end{array}$$

implies $[c_1,c_2,c_3]$. If points a_1, b_2, and c_3 have the common tangential, then points a_2, b_3, and c_1 have the common tangential, as do points a_3, b_1, and c_2.

Proof. Here, $[b_1,a_1,c_1]$ and $[c_2,b_2,a_2]$ imply $[b_1',a_1',c_1']$ and $[c_2',b_2',a_2']$, and since $[b_2,b_2,a_1']$ and $[c_3,c_3,b_2']$ also hold true, the tables

$$\begin{array}{cc|c|} \cline{3-3} b_1' & a_1' & c_1' \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ \cline{3-3} \end{array} \qquad \begin{array}{cc|c|} \cline{3-3} c_2' & b_2' & a_2' \\ c_2 & c_3 & c_1 \\ c_2 & c_3 & c_1' \\ \cline{3-3} \end{array}$$

imply $[b_3,b_3,c_1']$ and $[c_1,c_1,a_2']$ (i.e., $b_3' = c_1'$ and $c_1' = a_2'$, respectively). In the same way, the second claim follows from the first statement by cyclically replacing the indices $1 \to 2 \to 3 \to 1$. □

Theorem 8. *Let points a', b', and c' be tangentials of points a, b, and c. Then, $[a',b,c]$ and $[a,b',c]$ imply $[a,b,c']$.*

Proof. This statement follows from the table

$$\begin{array}{cc|c|} \cline{3-3} a' & a & a \\ b & b' & b \\ c & c & c' \\ \cline{3-3} \end{array}. \quad \square$$

The next theorem is the converse of Lemma 2.

Theorem 9. *If $[a,b,c]$ and $[c,d,e]$ hold, and if points a and b have the common tangential a', then there is a point f satisfying $[a,e,f]$ and $[b,d,f]$ (i.e., there exists the quadrilateral $\{a,d;\ b,e;\ c,f\}$).*

Proof. Let f be a point such that $[a,e,f]$. From the table

$$\begin{array}{cc|c|} \cline{3-3} a & c & b \\ a' & d & d \\ a & e & f \\ \cline{3-3} \end{array} \quad \square$$

we obtain $[b,d,f]$.

Theorem 10. *Let a, b, c, d, e, and f be points such that $[b,c,d]$, $[c,a,e]$, and $[a,b,f]$. The following four statements are equivalent: $a' = d'$, $b' = e'$, $c' = f'$, and $[d,e,f]$.*

Proof. Because of the symmetry of our ternary relation, it suffices to prove that $a' = d'$ if and only if $[d, e, f]$. If $a' = d'$, then $[d, e, f]$ follows from the table

$$\begin{array}{cc|c|} \hline a' & d & d \\ a & c & e \\ a & b & f \\ \hline \end{array}.$$

Conversely, if $[d, e, f]$, then $a' = d'$ follows from the table

$$\begin{array}{cc|c|} \hline c & b & d \\ e & f & d \\ a & a & a' \\ \hline \end{array}. \quad \square$$

The following theorem is the converse of Lemma 3.

Theorem 11. *If there exist lines $[a, b, p]$, $[c, d, p]$, $[a, c, q]$, $[b, d, q]$, $[a, d, r]$, and $[b, c, r]$, then points a, b, c, and d have the common tangential t, and points p, q, r, and t have the common tangential.*

Proof. If t is the tangential of point a, then the fact that point t is the common tangential of points b, c, and d follows from the tables

$$\begin{array}{cc|c|} \hline q & d & b \\ c & r & b \\ a & a & t \\ \hline \end{array}, \qquad \begin{array}{cc|c|} \hline p & d & c \\ b & r & c \\ a & a & t \\ \hline \end{array}, \qquad \begin{array}{cc|c|} \hline p & c & d \\ b & q & d \\ a & a & t \\ \hline \end{array}.$$

According to the table

$$\begin{array}{cc|c|} \hline a & b & p \\ a & b & p \\ t & t & t' \\ \hline \end{array},$$

the point t' is the tangential of p. The proof for points q and r is similar. $\square$

Theorem 12. *Let $[b, c, d]$, $[c, a, e]$, and $[a, b, f]$ hold. Points a, b, and c have the common tangential if and only if there exists a point p such that $[a, d, p]$, $[b, e, p]$, and $[c, f, p]$.*

Proof. Let $a' = b' = c' = q$, and let p be a point where $[a, d, p]$. Statements $[b, e, p]$ and $[c, f, p]$ follow from the tables

$$\begin{array}{cc|c|} \hline q & b & b \\ a & c & e \\ a & d & p \\ \hline \end{array} \qquad \begin{array}{cc|c|} \hline q & c & c \\ a & b & f \\ a & d & p \\ \hline \end{array}.$$

Conversely, if there is a point p such that $[a, d, p]$, $[b, e, p]$, and $[c, f, p]$, and if q is the tangential of the point a, then from the tables

$$\begin{array}{cc|c|} \hline p & e & b \\ d & c & b \\ a & a & q \\ \hline \end{array} \qquad \begin{array}{cc|c|} \hline p & f & c \\ d & b & c \\ a & a & q \\ \hline \end{array}$$

we find that q is also the tangential of b and c. $\square$

Theorem 13. *If $[b, c, d]$, $[c, a, e]$, and $[a, b, f]$, then there exists a point p satisfying $[a', d, p]$ and $[e, f, p]$.*

Proof. The proof follows by applying the table

$$\begin{array}{cc|c|}\cline{3-3} a & a & a' \\ c & b & d \\ e & f & p \\ \cline{3-3}\end{array}. \quad \square$$

Theorem 14. *If $[b,c,d]$, $[c,a,e]$, and $[a,b,f]$, then $[a',b',c']$ and $[d,e,f]$ are equivalent statements.*

Proof. Suppose $[a',b',c']$ holds. Since $[b,c,d]$ implies $[b',c',d']$, we find that $d' = a'$ (i.e., $[d,d,a']$). Due to Theorem 13, there is a point p such that $[a',d,p]$ and $[e,f,p]$, which imply $p = d$ (i.e., $[e,f,d]$ holds true). Conversely, suppose $[d,e,f]$ holds. From the table

$$\begin{array}{cc|c|}\cline{3-3} e & f & d \\ c & b & d \\ a & a & a' \\ \cline{3-3}\end{array},$$

we find that $d' = a'$. As $[b,c,d]$ implies $[b',c',d']$, we obtain $[b',c',a']$. $\square$

Following [10], a trio of points (a,b,c) is called a *triad* if for points d, e, and f such that $[b,c,d]$, $[c,a,e]$, and $[a,b,f]$, the statement $[d,e,f]$ also holds true (i.e., there is a quadrilateral $\{d,a;\ e,b;\ f,c\}$ in which the triad (a,b,c) is a triangle). We will call this quadrilateral the *circumscribed quadrilateral* and line $[d,e,f]$ the *complementary line* of the triad (a,b,c). Obviously, the quadrilateral $\{d,a;e,b;f,c\}$ is also circumscribed to three other triads (a,e,f), (b,d,f), and (c,d,e), to which the lines $[b,c,d]$, $[c,a,e]$, and $[a,b,f]$ are complementary. Due to Lemma 1, the pairs of points a,d; b,e; and c,f have the common tangentials a', b', and c' belonging to one line.

Theorem 15. *If $\{d,a;\ e,b;\ f,c\}$ is the circumscribed quadrilateral of the triad (a,b,c), and if g, h, and i are points such that $[a,d,g]$, $[b,e,h]$, and $[c,f,i]$ hold, then these points also form the triad (g,h,i) whose complementary line is the line (d',e',f'), where d', e', and f' are the tangentials of points d, e, and f.*

Proof. We already have $[b,c,d]$, $[c,a,e]$, $[a,b,f]$, and $[d,e,f]$, and the last relation implies $[d',e',f']$. From the tables

$$\begin{array}{cc|c|}\cline{3-3} b & e & h \\ c & f & i \\ d & d & d' \\ \cline{3-3}\end{array}, \qquad \begin{array}{cc|c|}\cline{3-3} a & d & g \\ c & f & i \\ e & e & e' \\ \cline{3-3}\end{array}, \quad \text{and} \quad \begin{array}{cc|c|}\cline{3-3} a & d & g \\ b & e & h \\ f & f & f' \\ \cline{3-3}\end{array},$$

we obtain $[h,i,d']$, $[g,i,e']$, and $[g,h,f']$ which, together with $[d',e',f']$, yields the statement of the theorem. $\square$

Theorem 16. *Let a_1, a_2, and a_3 be given points, let (i,j,k) be any cyclic permutation of $(1,2,3)$, and let points b_i and c_i $(i = 1,2,3)$ be defined in such a way that $[a_i,a_j,b_k]$ and $[b_i,b_j,c_k]$ hold. Then, $[b_i,c_i,a'_i]$ holds for $i = 1,2,3$.*

Proof. The statement follows using the table

$$\begin{array}{cc|c|}\cline{3-3} a_k & a_j & b_i \\ b_j & b_k & c_i \\ a_i & a_i & a'_i \\ \cline{3-3}\end{array}. \quad \square$$

3. Properties of the Tangentials in the Cubic Structures of Ranks 0 and 1

Theorem 17. *In a cubic structure of rank* 0*, the antecedents of three collinear points are also collinear.*

Proof. Let a, b, and c be antecedents of points a', b', and c', respectively, and let $[a', b', c']$ hold true. Suppose that there is a point d such that $[a, b, d]$. This implies $[a', b', d']$, and thus $d' = c'$. Since each point has only one antecedent, we conclude that $d = c$. □

Theorem 18. *In a cubic structure of rank* 1*, let a_1 and a_2 be two different points having the common tangential a, and let b be a point different from a which has the common tangential with a. Then, $[a_1, a_2, b]$ holds true.*

Proof. Let c be a point such that $[a_1, a_2, c]$, and let a' be the tangential of a. From the table

a_1	a_2	c
a_1	a_2	c
a	a	a'

it follows that point c has the tangential a'. If $c = a$, then we would have $[a_1, a_2, a]$ and $[a_1, a_1, a]$. This would lead to the contradiction $a_1 = a_2$. Consequently, $c = b$. □

Theorem 19. *If points a and b have the tangentials a' and b', respectively, and if point c' is such that $[a', b', c']$, then there exists a point c to which point c' is tangential and which satisfies $[a, b, c]$.*

Proof. Let c be a point such that $[a, b, c]$. This implies $[a', b', c']$, and the point c' is uniquely determined. □

Theorem 20. *In a cubic structure of rank* 1*, let $[a', b', c']$ hold true, and let a_1, a_2; b_1, b_2; and c_1, c_2 be pairs of different points with common tangentials a', b', and c', respectively. Then, the indices of these points can be chosen such that $[a_1, b_1, c_1]$, $[a_1, b_2, c_2]$, $[a_2, b_1, c_2]$, and $[a_2, b_2, c_1]$ (i.e., such that $\{a_1, a_2;\ b_1, b_2;\ c_1, c_2\}$ is a quadrilateral).*

Proof. Let us choose arbitrary labeled points with tangentials a' and b'. Due to Theorem 19, one of the points with the tangential c' lies on the same line with points a_1 and b_1. Let us label this point with c_1 and the other point with c_2. We therefore have $[a_1, b_1, c_1]$. Then, because of axiom C1, $[a_1, b_2, c_1]$ and $[a_2, b_1, c_1]$ cannot be valid. According to Theorem 19, $[a_1, b_2, c_2]$ and $[a_2, b_1, c_2]$ must hold. From any of these two statements, due to C1, it follows that $[a_2, b_2, c_2]$ cannot be valid, and according to Theorem 19, $[a_2, b_2, c_1]$ must hold. □

4. The Properties of Tangentials in Cubic Structures of Rank 2

Theorem 21. *If non-collinear points a, b, and c in a cubic structure of rank* 2 *have collinear tangentials a', b', and c', respectively, then (a, b, c) is a triad.*

Proof. Each of the points a', b', and c' has four antecedents a_i, b_i, and c_i $(i = 0, 1, 2, 3)$. According to Lemma 4, the indices of these points can be chosen such that, among others, $[a_0, b_0, c_0]$, $[a_0, b_1, c_1]$, $[a_1, b_0, c_1]$, and $[a_1, b_1, c_0]$ hold, where points a_1, b_1, and c_1 are the original points a, b, and c. □

Theorem 22. *Let the triad (a, b, c) and the line $[d, e, f]$ be given. If g, h, and i are points such that $[a, d, g]$, $[b, e, h]$, and $[c, f, i]$, then (g, h, i) is also a triad.*

Proof. Here, $[a', b', c']$ holds, and the existence of four lines in the theorem implies the existence of the lines $[d', e', f']$, $[a', d', g']$, $[b', e', h']$, and $[c', f', i']$. From the table

$$\begin{array}{cc|c|} \cline{3-3} a' & d' & g' \\ b' & e' & h' \\ c' & f' & i' \\ \cline{3-3} \end{array}$$

we obtain $[g', h', i']$. Points g, h, and i are not collinear, because otherwise $[a, b, c]$ would follow from the table

$$\begin{array}{cc|c|} \cline{3-3} d & g & a \\ e & h & b \\ f & i & c \\ \cline{3-3} \end{array},$$

which is not true. Due to Theorem 20, (g, h, i) is a triad. □

Theorem 23. *Let a, b, and c be three non-collinear points in a cubic structure of rank 2. Then, there are four triples of the form (d, e, f) such that $[e, f, a]$, $[f, d, b]$, and $[d, e, c]$ hold true.*

Proof. Let g be a point such that $[b, c, g]$, and additionally, let h be a point such that $[a, g, h]$. If we assume that $[e, f, a]$, $[f, d, b]$, and $[d, e, c]$ hold, then from the table

$$\begin{array}{cc|c|} \cline{3-3} f & b & d \\ e & c & d \\ a & g & h \\ \cline{3-3} \end{array}$$

it follows that point h is necessarily the tangential of d. Therefore, for point d, we have to take one of the four antecedents of h. Let us take one such point d, and then let e and f be points such that $[c, d, e]$ and $[b, d, f]$ hold. Then $[e, f, a]$ follows from the table

$$\begin{array}{cc|c|} \cline{3-3} d & c & e \\ d & b & f \\ h & g & a \\ \cline{3-3} \end{array}. \quad \square$$

In the previous inference, if points a, b, and c are collinear, then $g = a$, and thus $[a, a, h]$. Therefore, point a is one of the antecedents of point h, and any of the other three can be taken as point d. In case $d = a$, we obtain $e = b$ and $f = c$. Theorem 16 solves the problem of finding a triangle whose "sides" pass through the given points.

Theorem 24. *If a_1, a_2, a_3, and a_4 are different points with the common tangential a', then $[a_1, a_2, a_3, a_4, b, c]$ implies $[b, c, a'']$, where a'' is the tangential of a'.*

Proof. According to Lemma 2, there is a point p such that $[a_1, a_2, p]$, $[a_3, a_4, p]$, and points p an a' have the common tangential a''. Let t be a point such that $[b, c, t]$. As $[a_1, a_2, a_3, a_4, b, c]$, $[a_1, a_2, p]$, $[a_3, a_4, p]$, and $[b, c, t]$ imply $[p, p, t]$, we obtain that t is the tangential of p (i.e., $t = a''$). □

Let a, b, c, and d be associated points with the common tangential p. According to Lemma 3, there are points d, e, and f such that $[a, b, e]$, $[c, d, e]$, $[a, c, f]$, $[b, d, f]$, $[a, d, g]$, and $[b, c, g]$. Points p, e, f, and g are associated with the common tangential p'. According to the same lemma, there are points e_1, f_1, and g_1 such that $[p, e, e_1]$, $[f, g, e_1]$, $[p, f, f_1]$, $[e, g, f_1]$, $[p, g, g_1]$, and $[e, f, g_1]$. Points p', e_1, f_1, and g_1 are associated with the common tangential p''.

Theorem 25. *Using the same labeling and the results from above, there exist points q, r, s, and t such that $[p', a, q]$, $[b, e_1, q]$, $[c, f_1, q]$, $[d, g_1, q]$, $[p', b, r]$, $[a, e_1, r]$, $[d, f_1, r]$, $[c, g_1, r]$, $[p', c, s]$,*

$[d,e_1,s]$, $[a,f_1,s]$, $[b,g_1,s]$, $[p',d,t]$, $[c,e_1,t]$, $[b,f_1,t]$, and $[a,g_1,t]$ hold true, and points q, r, s, and t are associated.

Proof. Let point q be such that $[p',a,q]$. Then, $[b,e_1,q]$, $[c,f_1,q]$, and $[d,g_1,q]$ follow from the tables

$$\begin{array}{cc|c|} f & d & b \\ f & g & e_1 \\ p' & a & q \end{array}, \quad \begin{array}{cc|c|} e & d & c \\ e & g & f_1 \\ p' & a & q \end{array}, \quad \text{and} \quad \begin{array}{cc|c|} e & c & d \\ e & f & g_1 \\ p' & a & q \end{array}.$$

Similarly, $[p',b,r]$ implies $[a,e_1,r]$, $[d,f_1,r]$, and $[c,g_1,r]$, and from $[p',c,s]$, we obtain $[d,e_1,s]$, $[a,f_1,s]$, and $[b,g_1,s]$, and $[p',d,t]$ implies $[c,e_1,t]$, $[b,f_1,t]$, and $[a,g_1,t]$. Finally, switching to the tangentials, from $[p',a,q]$, $[p',b,r]$, $[p',c,s]$, and $[p',d,t]$, we obtain $[p'',p,q']$, $[p'',p,r']$, $[p'',p,s']$, and $[p'',p,t']$ (i.e., $q'=r'=s'=t'$). □

5. Hesse Configuration

The configuration $(12_3, 16_4)$ of points and lines obtained in Lemma 4 is called the *Hesse configuration*. In this configuration, we have the line $[a_1,b_1,c_1]$, nine lines of the forms $[a_1,b_i,c_i]$, $[a_i,b_1,c_i]$, and $[a_i,b_i,c_1]$ $(i=2,3,4)$, and six lines of the form $[a_i,b_j,c_k]$, where (i,j,k) is any permutation of $(2,3,4)$. The lines of this configuration can be divided into 4 quadruplets of lines, each of which contains all 12 points of the configuration:

$$\begin{array}{llll} [a_1,b_1,c_1], & [a_2,b_3,c_4], & [a_3,b_4,c_2], & [a_4,b_2,c_3], \\ [a_1,b_2,c_2], & [a_2,b_4,c_3], & [a_3,b_3,c_1], & [a_4,b_1,c_4], \\ [a_1,b_3,c_3], & [a_2,b_1,c_2], & [a_3,b_2,c_4], & [a_4,b_4,c_1], \\ [a_1,b_4,c_4], & [a_2,b_2,c_1], & [a_3,b_1,c_3], & [a_4,b_3,c_2]. \end{array}$$

Moreover, because of the following two theorems, the points of the Hesse configuration lie with some other points on some more lines.

Theorem 26. *Using the notation from Lemma 4, there are points x_i, y_i, and z_i $(i=2,3,4)$ such that there are 18 lines $[a_1,a_i,x_i]$, $[a_j,a_k,x_i]$, $[b_1,b_i,y_i]$, $[b_j,b_k,y_i]$, $[c_1,c_i,z_i]$, and $[c_j,c_k,z_i]$, where $i=2,3,4$, $j,k\neq i$, and $j<k$.*

Proof. Let x_i, y_i, and z_i $(i=2,3,4)$ be points such that $[a_1,a_i,x_i]$, $[b_1,b_i,y_i]$, and $[c_1,c_i,z_i]$ hold. From the table

$$\begin{array}{cc|c|} b_1 & c_j & a_j \\ c_1 & b_k & a_k \\ a_1 & a_i & x_i \end{array}$$

we obtain $[a_j,a_k,x_i]$. Cyclically permuting letters a, b, and c in the previous table proves $[b_j,b_k,y_i]$, and repeating it proves $[c_j,c_k,z_i]$. □

Theorem 27. *For each permutation (i,j,k) of $(2,3,4)$, for the points from Theorem 22 there is a line $[x_i,y_j,z_k]$.*

Proof. The proof follows by applying the table

$$\begin{array}{cc|c|} a_1 & a_i & x_i \\ b_1 & b_j & y_j \\ c_1 & c_k & z_k \end{array}. \quad \square$$

Theorem 28. *Let a' be the common tangential of mutually different points a_1, a_2, a_3, and a_4 in a cubic structure of rank 2. If o is any point, and if b_1, b_2, b_3, and b_4 are points such that $[o,a_i,b_i]$ $(i=1,2,3,4)$, then b_1, b_2, b_3, and b_4 are mutually different points with the common tangential b' such that $[o',a',b']$, where o' is the tangential of point o. In addition, there are mutually different points c, d, and e which are different from point o such that points c, d, e, and o*

have the common tangential o' and $[a_1, b_2, c]$, $[a_2, b_1, c]$, $[a_3, b_4, c]$, $[a_4, b_3, c]$, $[a_1, b_3, d]$, $[a_3, b_1, d]$, $[a_2, b_4, d]$, $[a_4, b_2, d]$, $[a_1, b_4, e]$, $[a_4, b_1, e]$, $[a_2, b_3, e]$, and $[a_3, b_2, e]$ hold true.

Proof. Let a_1 and a_2 be different points with the common tangential a', let o be any point, and let b_1 and b_2 be points such that $[o, a_1, b_1]$ and $[o, a_2, b_2]$. Then, according to Lemma 1, points b_1 and b_2 are different and have the common tangential b' such that $[o', a', b']$, where o' is the tangential of o. Aside from that, there is a point c such that $[a_1, b_2, c]$ and $[a_2, b_1, c]$, and points o and c have the common tangential o'. Points o and c are different because otherwise, we would have $[a_1, b_1, o]$ and $[a_1, b_2, o]$, where $b_i \neq b_2$. In a cubic structure of rank 2, each point has four different antecedent points. Let a_1, a_2, a_3, and a_4 be different points with the common tangential a'. If o is any point, and b_1, b_2, b_3, and b_4 are points such that $[o, a_i, b_i]$ $(i = 1, 2, 3, 4)$, then points b_1, b_2, b_3, and b_4 are mutually different. Due to the previous facts, points b_1, b_2, b_3, and b_4 have the common tangential b' such that $[o', a', b']$, and there are points c, d, and e such that $[a_1, b_2, c]$, $[a_2, b_1, c]$, $[a_1, b_3, d]$, $[a_3, b_1, d]$, $[a_1, b_4, e]$, and $[a_4, b_1, e]$ and which have the tangential o'. Points c, d, and e are mutually different because, for example, $c = d$ would imply $[a_1, b_3, c]$, which contradicts $[a_1, b_2, c]$, where $b_2 \neq b_3$, and they are also different from point o. There is also point f with tangential o' and different from o such that $[a_2, b_3, f]$ and $[a_3, b_2, f]$. It must coincide with one of points c, d, or e. Due to $[a_2, b_1, c]$ and $[a_1, b_3, d]$, it can be neither c nor d, and thus $f = e$ (i.e., $[a_2, b_3, e]$ and $[a_3, b_2, e]$ hold true). Similarly, one can show that $[a_2, b_4, d]$, $[a_4, b_2, d]$, $[a_3, b_4, c]$, and $[a_4, b_3, c]$. □

If we now rename points o, c, d, and e as c_0, c_1, c_2, and c_3, respectively, then we have 16 lines of the form $[a_i, b_j, c_k]$, where the indices $i, j, k \in \{0, 1, 2, 3\}$ are such that either all three are different or all three are equal to zero. Otherwise, one index is equal to zero, and the other two are equal and different from zero. We find the Table 1

Table 1. Hesse configuration $(12_4, 16_3)$ (of points a_i, b_i, c_i $(i = 0, 1, 2, 3)$).

	b_0	b_1	b_2	b_3
a_0	c_0	c_1	c_2	c_3
a_1	c_1	c_0	c_3	c_2
a_2	c_2	c_3	c_0	c_1
a_3	c_3	c_2	c_1	c_0

from which one can see which point c_k lies on the same line with some point a_i and some point b_j. We once again obtain the Hesse configuration $(12_4, 16_3)$ from Lemma 4.

Theorem 29. *The following quadrilaterals exist in the Hesse configuration shown in the previous table:*

$$\{a_0, a_1; b_0, b_1; c_0, c_1\}, \{a_0, a_1; b_2, b_3; c_2, c_3\}, \{a_2, a_3; b_0, b_1; c_2, c_3\}, \{a_2, a_3; b_2, b_3; c_0, c_1\},$$

$$\{a_0, a_2; b_0, b_2; c_0, c_2\}, \{a_0, a_2; b_1, b_3; c_1, c_3\}, \{a_1, a_3; b_0, b_2; c_1, c_3\}, \{a_1, a_3; b_1, b_3; c_0, c_2\},$$

$$\{a_0, a_3; b_0, b_3; c_0, c_3\}, \{a_0, a_3; b_1, b_2; c_1, c_2\}, \{a_1, a_2; b_0, b_3; c_1, c_2\}, \{a_1, a_2; b_1, b_2; c_0, c_3\}.$$

Proof. The proof is obvious when referring to the above table. For example, the last statement is a consequence of the existence of lines $[a_1, b_1, c_0]$, $[a_1, b_2, c_3]$, $[a_2, b_1, c_3]$, and $[a_2, b_2, c_0]$. □

6. The de Vries Configuration

In [11,12], it is claimed (and in [13], it is proven) that for an elliptic cubic, there are only two non-isomorphic configurations $(12_4, 16_3)$ in which there are three disjoint quadruples of points such that no two points from a particular quadruple are on one of the 16 lines of the configuration. One of these configurations is the Hesse configuration, and the other can be called the *de Vries configuration*. Both configurations were also observed in [14–16]. All five authors used the properties of the ambient space (i.e., the properties of the projective plane in which the cube is embedded). In this paper, we will observe the de Vries configuration in any cubic structure by means of that structure (i.e., using only axioms C1–C3). The observed cubic structure should be of at least rank 1.

We start from three non-collinear points a_0, a_1, and b_0. Let c_0 and a_{01} be points such that $[a_0, b_0, c_0]$ and $[a_0, a_1, a_{01}]$, and let a_2 and a_3 be two different points with the common tangential a_{01}. Let b_2, b_3, c_2, and c_3 be points such that $[c_0, a_2, b_3]$, $[c_0, a_3, b_2]$, $[b_0, a_2, c_2]$, and $[b_0, a_3, c_3]$ hold, and let b_1 be the point such that $[a_2, c_3, b_1]$. Then, by using the tables

$$\begin{array}{cc|c|}\cline{3-3} a_{01} & a_0 & a_1 \\ a_2 & c_0 & b_3 \\ a_2 & b_0 & c_2 \\ \cline{3-3}\end{array}\,, \qquad \begin{array}{cc|c|}\cline{3-3} a_{01} & a_0 & a_1 \\ a_3 & c_0 & b_2 \\ a_3 & b_0 & c_3 \\ \cline{3-3}\end{array}\,, \qquad \begin{array}{cc|c|}\cline{3-3} a_{01} & a_3 & a_3 \\ a_2 & b_0 & c_2 \\ a_2 & c_3 & b_1 \\ \cline{3-3}\end{array}\,.$$

we obtain $[a_1, b_3, c_2]$, $[a_1, b_2, c_3]$, and $[a_3, c_2, b_1]$. If b_1 and c_1 are points such that $[a_1, c_0, b_1]$ and $[a_1, b_0, c_1]$, then from the tables

$$\begin{array}{cc|c|}\cline{3-3} b_1 & c_3 & a_2 \\ c_0 & a_3 & b_2 \\ a_1 & b_0 & c_1 \\ \cline{3-3}\end{array}\,, \qquad \begin{array}{cc|c|}\cline{3-3} b_1 & c_2 & a_3 \\ c_0 & a_2 & b_3 \\ a_1 & b_0 & c_1 \\ \cline{3-3}\end{array}$$

we have $[a_2, b_2, c_1]$ and $[a_3, b_3, c_1]$. Finally, from the tables

$$\begin{array}{cc|c|}\cline{3-3} c_0 & b_0 & a_0 \\ a_3 & c_2 & b_1 \\ b_2 & a_2 & c_1 \\ \cline{3-3}\end{array}\,, \qquad \begin{array}{cc|c|}\cline{3-3} c_0 & b_0 & a_0 \\ a_2 & c_1 & b_2 \\ b_3 & a_1 & c_2 \\ \cline{3-3}\end{array}\,, \qquad \begin{array}{cc|c|}\cline{3-3} c_0 & b_0 & a_0 \\ a_1 & c_2 & b_3 \\ b_1 & a_2 & c_3 \\ \cline{3-3}\end{array}\,.$$

we obtain $[a_0, b_1, c_1]$, $[a_0, b_2, c_2]$, and $[a_0, b_3, c_3]$. Therefore, we proved the following theorem:

Theorem 30. *There exists a configuration* $(12_4, 16_3)$ *of points* a_i, b_i, *and* c_i, *where* $i = 0, 1, 2, 3$, *with the corresponding Table 2.*

Table 2. The de Vries configuration $(12_4, 16_3)$ (of points a_i, b_i, c_i $(i = 0, 1, 2, 3)$).

	b_0	b_1	b_2	b_3
a_0	c_0	c_1	c_2	c_3
a_1	c_1	c_0	c_3	c_2
a_2	c_2	c_3	c_1	c_0
a_3	c_3	c_2	c_0	c_1

The configuration from Theorem 30 is the de Vries configuration. To create it, we begin such that points a_2 and a_3 have the common tangential a_{01} satisfying $[a_0, a_1, a_{01}]$. However, such a property appears several times more in the configuration (i.e., the following holds true):

Theorem 31. *In the configuration from Theorem 30, pairs of points* a_0, a_1; a_2, a_3; b_0, b_1; b_2, b_3; c_0, c_1; *and* c_2, c_3 *have common tangentials* a_{23}, a_{01}, b_{23}, b_{01}, c_{23}, *and* c_{01}, *respectively, such that* $[a_2, a_3, a_{23}]$, $[a_0, a_1, a_{01}]$, $[b_2, b_3, b_{23}]$, $[b_0, b_1, b_{01}]$, $[c_2, c_3, c_{23}]$, *and* $[c_0, c_1, c_{01}]$ *hold true.*

Proof. We will prove all statements using only Table 2 and statements $[a_2, a_3, a_{23}]$, $[a_0, a_1, a_{01}]$, $[b_2, b_3, b_{23}]$, $[b_0, b_1, b_{01}]$, $[c_2, c_3, c_{23}]$, and $[c_0, c_1, c_{01}]$ (i.e., independent from the construction method of the observed configuration). Statements $[a_0, a_0, a_{23}]$, $[a_1, a_1, a_{23}]$, $[a_2, a_2, a_{01}]$, and $[a_3, a_3, a_{01}]$ can be derived from the tables

$$\begin{array}{cc|c|}\cline{3-3} b_0 & c_0 & a_0 \\ c_2 & b_2 & a_0 \\ a_2 & a_3 & a_{23} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} b_0 & c_1 & a_1 \\ c_2 & b_3 & a_1 \\ a_2 & a_3 & a_{23} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} b_0 & c_2 & a_2 \\ c_0 & b_3 & a_2 \\ a_0 & a_1 & a_{01} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} b_0 & c_3 & a_3 \\ c_0 & b_2 & a_3 \\ a_0 & a_1 & a_{01} \\ \cline{3-3}\end{array}\,.$$

We obtain $[b_0, b_0, b_{23}]$, $[b_1, b_1, b_{23}]$, $[b_2, b_2, b_{01}]$, and $[b_3, b_3, b_{01}]$ from the tables

$$\begin{array}{cc|c|}\cline{3-3} c_0 & a_0 & b_0 \\ a_3 & c_3 & b_0 \\ b_2 & b_3 & b_{23} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} c_0 & a_1 & b_1 \\ a_3 & c_2 & b_1 \\ b_2 & b_3 & b_{23} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} c_0 & a_3 & b_2 \\ a_0 & c_2 & b_2 \\ b_0 & b_1 & b_{01} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} c_0 & a_2 & b_3 \\ a_0 & c_3 & b_3 \\ b_0 & b_1 & b_{01} \\ \cline{3-3}\end{array}\,.$$

Then, $[c_0, c_0, c_{23}]$, $[c_1, c_1, c_{23}]$, $[c_2, c_2, c_{01}]$, and $[c_3, c_3, c_{01}]$ follow from the tables

$$\begin{array}{cc|c|}\cline{3-3} a_0 & b_0 & c_0 \\ b_2 & a_3 & c_0 \\ c_2 & c_3 & c_{23} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & b_1 & c_1 \\ b_2 & a_2 & c_1 \\ c_2 & c_3 & c_{23} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & b_2 & c_2 \\ b_0 & a_2 & c_2 \\ c_0 & c_1 & c_{01} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & b_3 & c_3 \\ b_0 & a_3 & c_3 \\ c_0 & c_1 & c_{01} \\ \cline{3-3}\end{array}\,. \quad \square$$

Theorem 32. *Under the conditions of Theorem 30, there exist complete quadrilaterals* $(a_0, a_1; b_0, b_1; c_0, c_1)$, $(a_2, a_3; b_2, b_3; c_1, c_0)$, $(a_2, a_3; b_0, b_1; c_2, c_3)$, *and* $(a_0, a_1; b_2, b_3; c_2, c_3)$.

Proof. Using Table 2, it is easy to check the existence of quadruples of the required lines. For instance, for the last quadrilateral, we have the lines $[a_0, b_2, c_2]$, $[a_0, b_3, c_3]$, $[a_1, b_2, c_3]$, and $[a_1, b_3, c_2]$. $\square$

From Table 2, we see that the lines

$$[a_0, b_0, c_0],\ [a_2, c_0, b_3],\ [a_0, b_3, c_3],\ [a_2, c_3, b_1],\ [a_0, b_1, c_1],\ [a_2, c_1, b_2],\ [a_0, b_2, c_2],\ [a_2, c_2, b_0]$$

pass alternately through points a_0 and a_2, while the lines

$$[a_1, b_0, c_1],\ [a_3, c_1, b_3],\ [a_1, b_3, c_2],\ [a_3, c_2, b_1],\ [a_1, b_1, c_0],\ [a_3, c_0, b_2],\ [a_1, b_2, c_3],\ [a_3, c_3, b_0]$$

pass alternately through points a_1 and a_3. In fact, we find two Steiner octagons with fundamental points a_0, a_2 and a_1, a_3, and all 16 lines of the configuration were used. However, it is also possible to form two Steiner octagons from all 16 lines of configuration with fundamental points a_0, a_3 and a_1, a_2, 2 with fundamental points b_0, b_2 and b_1 , b_3, 2 with fundamental points b_0, b_3 and b_1, b_2, 2 with fundamental points c_0, c_2 and c_1, c_3, and 2 with fundamental points c_0, c_3 and c_1, c_2. In the two observed octagons, opposing vertices have the common tangential, and the same holds for the other Steiner octagons.

Using the following theorems, we will show how to associate yet another de Vries configuration to the one from Theorem 30.

Theorem 33. *There exists a complete quadrilateral* $(a_{23}, a_{01}; b_{23}, b_{01}; c_{23}, c_{01})$.

Proof. From the tables

$$\begin{array}{cc|c|}\cline{3-3} a_3 & a_2 & a_{23} \\ b_3 & b_2 & b_{23} \\ c_1 & c_1 & c_{23} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_3 & a_2 & a_{23} \\ b_1 & b_0 & b_{01} \\ c_2 & c_2 & c_{01} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} b_3 & b_2 & b_{23} \\ c_1 & c_0 & c_{01} \\ a_3 & a_3 & a_{01} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} c_3 & c_2 & c_{23} \\ a_1 & a_0 & a_{01} \\ b_2 & b_2 & b_{01} \\ \cline{3-3}\end{array}$$

we obtain the lines $[a_{23}, b_{23}, c_{23}]$, $[a_{23}, b_{01}, c_{01}]$, $[a_{01}, b_{23}, c_{01}]$, and $[a_{01}, b_{01}, c_{23}]$, which proves the statement. $\square$

Since the opposite vertices of the quadrilaterals have common tangentials, we have the following:

Corollary 1. *The pairs of points a_{23}, a_{01}; b_{23}, b_{01}; and c_{23}, c_{01} have common tangentials.*

Theorem 34. *There exist points x_2, x_3, y_2, y_3, z_2, and z_3 such that there exist lines $[a_0, a_2, x_2]$, $[a_1, a_3, x_2]$, $[a_0, a_3, x_3]$, $[a_1, a_2, x_3]$, $[b_0, b_2, y_2]$, $[b_1, b_3, y_2]$, $[b_0, b_3, y_3]$, $[b_1, b_2, y_3]$, $[c_0, c_2, z_2]$, $[c_1, c_3, z_2]$, $[c_0, c_3, z_3]$, and $[c_1, c_2, z_3]$. Furthermore, there exist quadrilaterals $(a_0, a_1; a_2, a_3; x_2, x_3)$, $(b_0, b_1; b_2, b_3; y_2, y_3)$, and $(c_0, c_1; c_2, c_3; z_2, z_3)$. Finally, the pairs of points x_2, x_3; y_2, y_3; and z_2, z_3 have common tangentials.*

Proof. Let x_2, x_3, y_2, y_3, z_2, and z_3 be such that there are lines $[a_0, a_2, x_2]$, $[a_0, a_3, x_3]$, $[b_0, b_2, y_2]$, $[b_0, b_3, y_3]$, $[c_0, c_2, z_2]$, and $[c_0, c_3, z_3]$. The existence of the remaining six lines can be inferred from the following tables:

$$\begin{array}{cc|c|}\cline{3-3} c_0 & b_1 & a_1 \\ b_0 & c_3 & a_3 \\ a_0 & a_2 & x_2 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} b_0 & c_1 & a_1 \\ c_0 & b_3 & a_2 \\ a_0 & a_3 & x_3 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & c_1 & b_1 \\ c_0 & a_2 & b_3 \\ b_0 & b_2 & y_2 \\ \cline{3-3}\end{array}\,,$$

$$\begin{array}{cc|c|}\cline{3-3} a_0 & c_1 & b_1 \\ c_0 & a_3 & b_2 \\ b_0 & b_3 & y_3 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & b_1 & c_1 \\ b_0 & a_3 & c_3 \\ c_0 & c_2 & z_2 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & b_1 & c_1 \\ b_0 & a_2 & c_2 \\ c_0 & c_3 & z_3 \\ \cline{3-3}\end{array}\,.$$

The existence of these 12 lines proves the existence of the three mentioned quadrilaterals, and the last statement is an immediate consequence of [5] (Theorem 3.4). □

Theorem 35. *There exists a de Vries configuration $(12_4, 16_3)$ of points a_{01}, a_{23}, x_2, x_3, b_{01}, b_{23}, y_2 , y_3, c_{01}, c_{23}, z_2, and z_3 with the corresponding Table 3.*

Table 3. The de Vries configuration $(12_4, 16_3)$ (of points $a_{01}, a_{23}, x_2, x_3, b_{01}, b_{23}, y_2, y_3, c_{01}, c_{23}, z_2, z_3$).

	b_{01}	b_{23}	y_2	y_3
a_{01}	c_{23}	c_{01}	z_3	z_2
a_{23}	c_{01}	c_{23}	z_2	z_3
x_2	z_3	z_2	c_{01}	c_{23}
x_3	z_2	z_3	c_{23}	c_{01}

Proof. Except for the lines from Theorem 33, one should prove the existence of yet another 12 lines. But this follows from the following tables:

$$\begin{array}{cc|c|}\cline{3-3} a_0 & a_1 & a_{01} \\ b_0 & b_2 & y_2 \\ c_0 & c_3 & z_3 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & a_1 & a_{01} \\ b_0 & b_3 & y_3 \\ c_0 & c_2 & z_2 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_2 & a_3 & a_{23} \\ b_0 & b_2 & y_2 \\ c_2 & c_0 & z_2 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_2 & a_3 & a_{23} \\ b_0 & b_3 & y_3 \\ c_2 & c_1 & z_3 \\ \cline{3-3}\end{array}\,,$$

$$\begin{array}{cc|c|}\cline{3-3} a_0 & a_2 & x_2 \\ b_0 & b_1 & b_{01} \\ c_0 & c_3 & z_3 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & a_2 & x_2 \\ b_2 & b_3 & b_{23} \\ c_2 & c_0 & z_2 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & a_3 & x_3 \\ b_0 & b_1 & b_{01} \\ c_0 & c_2 & z_2 \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & a_3 & x_3 \\ b_2 & b_3 & b_{23} \\ c_2 & c_1 & z_3 \\ \cline{3-3}\end{array}\,,$$

$$\begin{array}{cc|c|}\cline{3-3} a_0 & a_2 & x_2 \\ b_0 & b_2 & y_2 \\ c_0 & c_1 & c_{01} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & a_2 & x_2 \\ b_2 & b_1 & y_3 \\ c_2 & c_3 & c_{23} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & a_3 & x_3 \\ b_2 & b_0 & y_2 \\ c_2 & c_3 & c_{23} \\ \cline{3-3}\end{array}\,, \quad \begin{array}{cc|c|}\cline{3-3} a_0 & a_3 & x_3 \\ b_0 & b_3 & y_3 \\ c_0 & c_1 & c_{01} \\ \cline{3-3}\end{array}\,.$$

Comparing Table 3 in the text of this theorem with Table 2 in Theorem 30 reveals that they represent the same configuration. □

Using Table 2, we proved Theorems 31 and 32 concerning the existence of some common tangentials and quadrilaterals. Similarly, by employing Table 3, we can prove analogous theorems about the existence of common tangentials and quadrilaterals in this second de Vries configuration.

7. Conclusions

The concept of a tangential in a general cubic structure was introduced and studied in [5]. In this paper, we explored the noteworthy properties of tangentials in cubic structures of ranks 0, 1, and 2. We investigated the relationships between tangentials and various other concepts in cubic structures of specific ranks. Additionally, we constructed the Hesse configuration of points and lines in a cubic structure of rank 2. We obtained and explored the de Vries configuration of points and lines in a cubic structure. The authors' future research aims to conduct a more detailed investigation into admissible and non-admissible configurations in cubic structures. However, in order to accomplish this, it is important to first dig up some additional significant properties of the tangentials in cubic structures beyond those already discovered in [5]. This paper used the cubic structure to demonstrate how the results can be reached with this quite simple structure. These findings were expressed in the language of models in the most well-known cubic structure: the geometry on cubic curves. However, certain additional cubic structure models were considered in [4], and therefore the findings achieved using a cubic structure were also readily obtained in these models.

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References

1. Etherington, L.M.H. Quasigroups and cubic curves. *Proc. Edinb. Math. Soc.* **1965**, *14*, 273–291.
2. Khalid, M. Group law on the cubic curve. *Irish Math. Soc. Bull.* **2007**, *60*, 67–89.
3. Zirilli, F. C-struttura associata ad una cubica piana e C-struttura astratta. *Ricerche Mat.* **1967**, *16*, 202–232.
4. Volenec, V.; Kolar-Begović, Z.; Kolar-Šuper, R. Cubic structure. *Glas. Mat. Ser. III* **2017**, *52*, 247–256.
5. Volenec, V.; Kolar-Begović, Z.; Kolar-Šuper, R. Tangentials in cubic structures. *Glas. Mat. Ser. III* **2020**, *55*, 337–349.
6. Cremona, L. *Einleitung in Eine Geometrische Theorie der Ebenen Kurven*; Hansebooks: Greifswald, Germany, 1865.
7. Durège, H. *Die Ebenen Curven Dritter Ordnung*; Teubner: Leipzig, Germany, 1871.
8. Schroeter, H. *Die Theorie der Ebenen Kurven Dritter Ordnung. Auf Synthetisch-Geometrischem Wege*; Teubner: Leipzig, Germany, 1888.
9. Smogorzhevski, A.S.; Stolova, E.S. *Handbook in the theory of plane curves of the third order*; State Publishing House of Physical and Mathematical Literature: Moscow, Russia, 1961. (In Russian)
10. Green, H.G.; Prior, L.E. Systems of triadic points on a cubic. *Amer. Math. Mon.* **1934**, *41*, 253–255.
11. de Vries, J. Über gewisse Configurationen auf ebenen kubischen Curven. *Sitzungsber. Akad. Wiss. Wien* **1889**, *98*, 1290–1298.
12. de Vries, J. Over vlakke configurations. *Versl. Kon. Akad. Wetensch.* **1889**, *5*, 105–120.
13. Schroeter, H. Die Hessesche Configuration $(12_4, 16_3)$. *J. Reine Angew. Math.* **1891**, *108*, 269–312.
14. Bydžovský, B. Poznámky k theorii konfigurace $(12_4, 16_3)$. *Čas. Pěst. Mat. Fys.* **1949**, *74*, 249–251.

15. Martinetti, V. Sopra un gruppo di configurazioni contenute nell' esagramma di Pascal. *Atti Accad. Gioenia Sci. Nat. Catania* **1890**, *3*, 31–50.
16. Zacharias, M. Untersuchungen über ebene Konfigurationen $(12_4, 16_3)$. *Deutsche Math.* **1941**, *6*, 147–170.