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Study of asymptotic behavior of solutions of neutral mixed type difference equations

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Abstract: In this paper, we consider a neutral mixed type difference equation, and obtain explicitly sufficient conditions for asymptotic behavior of solutions. A necessary condition is provided as well. An example is given to illustrate our main results.

Keywords: Contraction mapping, neutral difference equations, mixed type, asymptotic behavior.

MSC: 34K20, 34K30, 34D04.

1. Introduction

Certainly, the Lyapunov direct method has been, for more than 100 years, the efficient tool for the study of stability properties of ordinary, functional, partial differential and difference equations. Nevertheless, the application of this method to problems of stability in differential and difference equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded terms ([1–16]). Recently, Burton, Furumochi, Zhang, Raffoul, Islam, Yankson and others have noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [17–32]). The fixed point theory does not only solve the problem on stability but has a significant advantage over Lyapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise (see [1]).

In this paper, we consider the following mixed type neutral difference equation

$$\Delta x(t) + a(t)\Delta x(\tau(t)) + \sum_{i=1}^{k} b_i(t)\, x(\sigma_i(t)) + \sum_{j=1}^{l} c_j(t)\, x(\tau_j(t)) = 0, \tag{1}$$

with an assumed initial condition

$$x(t) = \psi(t) \text{ for } t \in [m(t_0), t_0] \cap \mathbb{Z}, \tag{2}$$

where $\psi : [m(t_0), t_0] \cap \mathbb{Z} \to \mathbb{R}$ is a bounded sequence and for $t_0 \geq 0$

$$m(t_0) = \inf\{\sigma_i(s) : s \geq t_0,\ i = 1, ...k\}.$$

Here Δ denotes the forward difference operator $\Delta x(t) = x(t+1) - x(t)$ for any sequence $\{x(t),\ t \in \mathbb{Z}^+\}$. For more details on the calculus of difference equations, we refer the reader to [11] and [24]. Throughout this paper, we assume that a, b_i and c_j are bounded sequences, and τ, σ_i and τ_j are non-negative sequences such that

$$\begin{aligned}
\tau(t) &\to \infty \text{ as } t \to \infty,\ \tau(t) \geq t,\ t \geq t_0,\\
\sigma_i(t) &\to \infty \text{ as } t \to \infty,\ i = 1, ..., k,\ \sigma_i(t) \leq t,\ t \geq t_0,\\
\tau_j(t) &\to \infty \text{ as } t \to \infty,\ j = 1, ..., l,\ \tau_j(t) \geq t,\ t \geq t_0.
\end{aligned}$$

Equation (1) can be viewed as a discrete analogue of the mixed type neutral differential equation;

$$x'(t) + a(t)\, x'(\tau(t)) + \sum_{i=1}^{k} b_i(t)\, x(\sigma_i(t)) + \sum_{j=1}^{l} c_j(t)\, x(\tau_j(t)) = 0. \tag{3}$$

In [25], Bicer investigated (3) and obtained the asymptotic behavior of solutions. Our purpose here is to show the asymptotic behavior of solutions for (1). An asymptotic stability theorem with a necessary and sufficient condition is proved by using the contraction mapping theorem. For details on contraction mapping principle we refer the reader to [33] . An example is given to illustrate our main results.

2. Main results

Theorem 1. *Let a, b_i and c_j non positive sequences. Assume that the following inequality has a nonnegative solution*

$$-a(t)\,\lambda(\tau(t)) \prod_{u=t}^{\tau(t)-1} (1-\lambda(u)) - \sum_{i=1}^{k} b_i(t) \prod_{u=t}^{\sigma_i(t)-1} (1-\lambda(u)) - \sum_{j=1}^{l} c_j(t) \prod_{u=t}^{\tau_j(t)-1} (1-\lambda(u)) \le \lambda(t)\,,\ t \ge t_0,$$

with $\lambda(t) < 1$. Then, (1) has a positive solution.

Proof. Let λ_0 be a nonnegative solution of (1). Set

$$\lambda_n(t) = \begin{cases} \lambda_{n-1}(t)\,, \quad \text{if } m(t_0) \le t \le t_0, \\ -a(t)\,\lambda_{n-1}(\tau(t)) \prod\limits_{u=t}^{\tau(t)-1} (1-\lambda_{n-1}(u)) - \sum\limits_{i=1}^{k} b_i(t) \prod\limits_{u=t}^{\sigma_i(t)-1} (1-\lambda_{n-1}(u)) \\ -\sum\limits_{j=1}^{l} c_j(t) \prod\limits_{u=t}^{\tau_j(t)-1} (1-\lambda_{n-1}(u))\,,\ t \ge t_0, \end{cases}$$

for $n = 1, 2, \ldots$. Then, by (1), we get

$$\lambda_0(t) \ge -a(t)\,\lambda_0(\tau(t)) \prod_{u=t}^{\tau(t)-1} (1-\lambda_0(u)) - \sum_{i=1}^{k} b_i(t) \prod_{u=t}^{\sigma_i(t)-1} (1-\lambda_0(u)) - \sum_{j=1}^{l} c_j(t) \prod_{u=t}^{\tau_j(t)-1} (1-\lambda_0(u)) = \lambda_1(t)\,.$$

Then, we obtain $\lambda_0(t) \ge \lambda_1(t) \ge \ldots \ge \lambda_n(t) \ge 0$. So, there exists a pointwise limit $\lambda(t) = \lim\limits_{n\to\infty} \lambda_n(t)$. So, from the Lebesgue convergence theorem, we obtain

$$\lambda(t) = -a(t)\,\lambda(\tau(t)) \prod_{u=t}^{\tau(t)-1} (1-\lambda(u)) - \sum_{i=1}^{k} b_i(t) \prod_{u=t}^{\sigma_i(t)-1} (1-\lambda(u)) - \sum_{j=1}^{l} c_j(t) \prod_{u=t}^{\tau_j(t)-1} (1-\lambda(u))\,.$$

Hence,

$$x(t) = \begin{cases} \lambda(t)\,, \quad \text{if } m(t_0) \le t \le t_0, \\ \lambda(t_0) \prod\limits_{u=t_0}^{t-1} (1-\lambda(u))\,,\ t \ge t_0, \end{cases}$$

is a positive solution of (1). □

Theorem 2. *Let a, b_i and c_j be non positive sequences and let $\Delta a(t) > 0$, $a(t_0) \ne -\infty$. If*

$$\sum_{u=t_0}^{\infty} \sum_{j=1}^{l} c_j(u) = -\infty,$$

and x is a eventually positive solution of (1), then $x(t) \to \infty$ as $t \to \infty$.

Proof. Assume that $x(t) > 0$ for $t \geq T_1$. Choose $T \geq T_1$ such that $T_1 \leq \inf\{\sigma_i(s) : s \geq T,\ i = 1, ..., k\}$. Then $\Delta x(t) + a(t)\Delta x(\tau(t)) \geq 0$, for $t \geq T$,

$$\Delta x(t) + a(t)\Delta x(\tau(t)) = -\sum_{i=1}^{k} b_i(t)\, x(\sigma_i(t)) - \sum_{j=1}^{l} c_j(t)\, x(\tau_j(t)),$$

and

$$\Delta [a(t)x(\tau(t))] = a(t)\Delta x(\tau(t)) + \Delta a(t)\, x(\tau(t+1)),$$

that is

$$\Delta [x(t) + a(t)x(\tau(t))] - \Delta a(t)\, x(\tau(t+1)) \geq -\sum_{j=1}^{l} c_j(t)\, x(\tau_j(t)).$$

From this, we can write

$$\Delta [x(t) + a(t)x(\tau(t))] \geq -\sum_{j=1}^{l} c_j(t)\, x(\tau_j(t)),$$

so

$$\Delta [x(t) + a(t)x(\tau(t))] \geq -x(T)\sum_{j=1}^{l} c_j(t),$$

which implies

$$x(t) + a(t)x(\tau(t)) \geq a(t_0)x(\tau(t_0)) - x(T)\sum_{u=t_0}^{t-1}\sum_{j=1}^{l} c_j(u).$$

So, we get

$$x(t) \geq a(t_0)x(\tau(t_0)) - x(T)\sum_{u=t_0}^{t-1}\sum_{j=1}^{l} c_j(u).$$

Then $x(t) \to \infty$ as $t \to \infty$. □

Theorem 3. *Let $a(t) > 0$, b_i and c_j be nonnegative sequences and let $\Delta a(t) < 0$, $a(t_0) \neq \infty$. If*

$$\sum_{u=t_0}^{\infty}\sum_{j=1}^{l} c_j(u) = \infty,$$

and x is a eventually positive solution of (1), then $x(t) \to 0$ as $t \to \infty$.

Proof. For $t \geq T_1$, since $x(t) > 0$ we Choose $T \geq T_1$ such that $T_1 \leq \inf\{\sigma_i(s) : s \geq T,\ i = 1, ..., k\}$. Then $\Delta x(t) + a(t)\Delta x(\tau(t)) \leq 0$, for $t \geq T$, and

$$\Delta x(t) + a(t)\Delta x(\tau(t)) \leq -\sum_{j=1}^{l} c_j(t)\, x(\tau_j(t)),$$

that is

$$\Delta [x(t) + a(t)x(\tau(t))] - \Delta a(t)\, x(\tau(t+1)) \leq -\sum_{j=1}^{l} c_j(t)\, x(\tau_j(t)).$$

From this, we can write

$$\Delta [x(t) + a(t)x(\tau(t))] \leq -\sum_{j=1}^{l} c_j(t)\, x(\tau_j(t)),$$

so

$$\Delta\left[x\left(t\right)+a(t)x\left(\tau\left(t\right)\right)\right]\leq -x\left(T\right)\sum_{j=1}^{l}c_{j}\left(t\right),$$

which implies

$$x\left(t\right)+a(t)x\left(\tau\left(t\right)\right)\leq a(t_{0})x\left(\tau\left(t_{0}\right)\right)-x\left(T\right)\sum_{u=t_{0}}^{t-1}\sum_{j=1}^{l}c_{j}\left(u\right).$$

So, we get

$$x\left(t\right)\leq a(t_{0})x\left(\tau\left(t_{0}\right)\right)-x\left(T\right)\sum_{u=t_{0}}^{t-1}\sum_{j=1}^{l}c_{j}\left(u\right).$$

Since $x(t)>0$, we get a contradiction.Then $x(t)\rightarrow 0$ as $t\rightarrow\infty$. □

Now, we investigate the asymptotic behavior of solutions of (1), free of the sign of the coefficients. During the process of inverting (1), an summation by parts will have to performed on the term involving $\Delta x(\tau(t))$.

Lemma 1. *A sequence x is a solution of (1)–(2) if and only if*

$$\begin{aligned} x\left(t\right) &= \left(x\left(t_{0}\right)+a\left(t_{0}-1\right)x\left(\tau\left(t_{0}\right)\right)\right)\prod_{s=t_{0}}^{t-1}\left(1-B(s)\right)-a(t-1)x(\tau\left(t\right)) \\ &+\sum_{r=t_{0}}^{t-1}\prod_{s=r+1}^{t-1}\left(1-B(s)\right)h\left(r\right)x\left(\tau\left(r\right)\right)-\sum_{r=t_{0}}^{t-1}\prod_{s=r+1}^{t-1}\left(1-B(s)\right)B\left(r\right)x\left(r+1\right) \\ &-\sum_{r=t_{0}}^{t-1}\prod_{s=r+1}^{t-1}\left(1-B(s)\right)\sum_{i=1}^{k}b_{i}\left(r\right)x\left(\sigma_{i}\left(r\right)\right)-\sum_{r=t_{0}}^{t-1}\prod_{s=r+1}^{t-1}\left(1-B(s)\right)\sum_{j=1}^{l}c_{j}\left(r\right)x\left(\tau_{j}\left(r\right)\right), \end{aligned} \tag{4}$$

for $t\geq t_{0}$, where

$$B\left(t\right)=\sum_{i=1}^{k}b_{i}\left(t\right)+\sum_{j=1}^{l}c_{j}\left(t\right),\ 0<B\left(t\right)<1,$$

and

$$h\left(t\right)=a\left(t\right)-a\left(t-1\right)\left(1-B(t)\right). \tag{5}$$

Proof. Since

$$x\left(\tau_{j}\left(t\right)\right)=x\left(t+1\right)+\sum_{u=t+1}^{\tau_{j}(t)-1}\Delta x\left(u\right),\ \text{and}\ x\left(\sigma_{i}\left(t\right)\right)=x\left(t+1\right)+\sum_{u=t+1}^{\sigma_{i}(t)-1}\Delta x\left(u\right).$$

We can rewrite (1) as

$$\Delta x\left(t\right)=-a\left(t\right)\Delta x\left(\tau\left(t\right)\right)-\sum_{i=1}^{k}b_{i}\left(t\right)\sum_{u=t+1}^{\sigma_{i}(t)-1}\Delta x\left(u\right)-\sum_{j=1}^{l}c_{j}\left(t\right)\sum_{u=t+1}^{\tau_{j}(t)-1}\Delta x\left(u\right)-B\left(t\right)x\left(t+1\right). \tag{6}$$

Multiplying both sides of (6) with $\prod_{s=t_{0}}^{t}\left(1-B(s)\right)^{-1}$, by summing from t_{0} to $t-1$, we obtain

$$\begin{aligned} \sum_{r=t_{0}}^{t-1}\Delta\left[\prod_{s=t_{0}}^{r-1}\left(1-B(s)\right)^{-1}x\left(r\right)\right] &= -\sum_{r=t_{0}}^{t-1}\prod_{s=t_{0}}^{r}\left(1-B(s)\right)^{-1}a\left(r\right)\Delta x\left(\tau\left(r\right)\right)-\sum_{r=t_{0}}^{t-1}\prod_{s=t_{0}}^{r}\left(1-B(s)\right)^{-1}B\left(r\right)x\left(r+1\right) \\ &-\sum_{r=t_{0}}^{t-1}\prod_{s=t_{0}}^{r}\left(1-B(s)\right)^{-1}\sum_{i=1}^{k}b_{i}\left(t\right)\sum_{u=t+1}^{\sigma_{i}(r)-1}\Delta x\left(u\right)-\sum_{r=t_{0}}^{t-1}\prod_{s=t_{0}}^{r}\left(1-B(s)\right)^{-1}\sum_{j=1}^{l}c_{j}\left(t\right)\sum_{u=t+1}^{\tau_{j}(r)}\Delta x\left(u\right). \end{aligned}$$

By dividing both sides of the above expression by $\prod_{s=t_0}^{t-1}(1-B(s))^{-1}$ we get

$$\begin{aligned} x(t) &= x(t_0)\prod_{s=t_0}^{t-1}(1-B(s)) - \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))a(r)\,\Delta x(\tau(r)) - \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\,B(r)\,x(r+1) \\ &\quad - \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\sum_{i=1}^{k}b_i(r)\sum_{u=t+1}^{\sigma_i(r)-1}\Delta x(u) - \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\sum_{j=1}^{l}c_j(r)\sum_{u=t+1}^{\tau_j(r)-1}\Delta x(u). \end{aligned} \tag{7}$$

By performing an summation by parts, we get

$$\begin{aligned} &\sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))a(r)\,\Delta x(\tau(r)) \\ &= a(t-1)\,x(\tau(t)) - a(t_0-1)\,x(\tau(t_0))\prod_{s=t_0}^{t-1}(1-B(s)) - \sum_{r=t_0}^{t-1}\Delta\left[\prod_{s=r}^{t-1}(1-B(s))a(r-1)\right]x(\tau(r)). \end{aligned} \tag{8}$$

But,

$$\begin{aligned} \sum_{r=t_0}^{t-1}\Delta\left[\prod_{s=r}^{t-1}(1-B(s))a(r-1)\right]x(\tau(r)) &= \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\,[a(r)-a(r-1)(1-B(r))]\,x(\tau(r)) \\ &= \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))h \end{aligned}$$

where h is given by (5). We obtain (4) by replacing (8) into (7). Since each step is reversible, the converse follows easily. This completes the proof. □

Theorem 4. *Assume that $0 < B(t) < 1$ and the following conditions hold*

$$\prod_{s=t_0}^{t-1}(1-B(s)) \to 0 \text{ as } t \to \infty, \tag{9}$$

and

$$\begin{aligned} &|a(t-1)| + \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\,|h(r)| + \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\,|B(r)| \\ &+ \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\sum_{i=1}^{k}|b_i(r)| + \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\sum_{j=1}^{l}|c_j(r)| \le \beta < 1. \end{aligned} \tag{10}$$

Then for each initial condition (2), every solution of (1) converges to zero.

Proof. Let $x \in C([m(t_0),\infty)\cap\mathbb{Z})$ is the space of all bounded sequences and $M = \{x \in C([m(t_0),\infty)\cap\mathbb{Z}) : x(t) \to 0 \text{ as } t \to \infty\}$, be a closed subspace. Then $(M,\|.\|)$ is a Banach space with the norm $\|x\| = \sup_{t\ge m(t_0)}|x(t)|$.

Define the operator $\phi : M \to M$ by

$$(\phi x)(t) = \begin{cases} \psi(t), \quad \text{if } m(t_0) \le t \le t_0, \\ (x(t_0) + a(t_0-1)\,x(\tau(t_0)))\prod_{s=t_0}^{t-1}(1-B(s)) - a(t-1)x(\tau(t)) \\ + \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\,h(r)\,x(\tau(r)) - \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\,B(r)\,x(r+1) \\ - \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\sum_{i=1}^{k}b_i(r)\,x(\sigma_i(r)) - \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\sum_{j=1}^{l}c_j(r)\,x(\tau_j(r)), \; t \ge t_0. \end{cases} \tag{11}$$

It is clear that for $x \in M$, ϕx is bounded. Now, we will show that ϕ is a contraction. Let x and y be two bounded sequences on $[m(t_0), \infty) \cap \mathbb{Z}$ and satisfying same initial condition (2). Then for $t \geq t_0$, we get

$$|(\phi x)(t) - (\phi y)(t)| \leq |a(t-1)| |x(\tau(t)) - y(\tau(t))| + \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) |h(r)| |x(\tau(r)) - y(\tau(r))|$$
$$+ \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) |x(r+1) - y(r+1)| |B(r)| + \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) \sum_{i=1}^{k} |b_i(r)| |x(\sigma_i(r)) - y(\sigma_i(r))|$$
$$+ \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) \sum_{j=1}^{l} |c_j(r)| \left|x(\tau_j(r)) - y(\tau_j(r))\right| \leq \beta \|x - y\|.$$

Thus, the operator ϕ has a unique fixed point in M, which solves (1). Now, we will show that, $(\phi x)(t) \to 0$ as $t \to \infty$. Actually, for $x \in M$, we have

$$\begin{aligned} |(\phi x)(t)| \leq\ & |(x(t_0) + a(t_0 - 1) x(\tau(t_0)))| \prod_{s=t_0}^{t-1} (1 - B(s)) + |a(t-1)| |x(\tau(t))| \\ & + \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) |h(r)| |x(\tau(r))| + \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) |x(r+1)| |B(r)| \\ & + \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) \sum_{i=1}^{k} |b_i(r)| |x(\sigma_i(r))| + \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) \sum_{j=1}^{l} \left|c_j(r)\right| \left|x(\tau_j(r))\right|. \end{aligned} \quad (12)$$

Note that by (9),

$$|(x(t_0) + a(t_0 - 1) x(\tau(t_0)))| \prod_{s=t_0}^{t-1} (1 - B(s)) \to 0 \text{ as } t \to \infty.$$

Moreover, since $x(t) \to 0$ as $t \to \infty$, for each $\varepsilon > 0$, there exists $T_1 > t_0$ such that $u \geq T_1$ implies that $|x(\tau(u))| < \frac{\varepsilon}{2}$. Thus, for $t \geq T_1$, the third term I_3 in (12) satisfies

$$\begin{aligned} I_3 &= \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) |h(r)| |x(\tau(r))| \\ &\leq \sum_{r=t_0}^{T_1-1} \prod_{s=r+1}^{t-1} (1 - B(s)) |h(r)| |x(\tau(r))| + \sum_{r=T_1}^{t-1} \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) |h(r)| |x(\tau(r))| \\ &\leq \sum_{r=t_0}^{T_1-1} \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) |h(r)| |x(\tau(r))| + \frac{\varepsilon}{2} \sum_{r=T_1}^{t-1} \sum_{r=t_0}^{t-1} \prod_{s=r+1}^{t-1} (1 - B(s)) |h(r)| \\ &\leq \frac{\varepsilon}{2} + \beta \frac{\varepsilon}{2} \\ &\leq \varepsilon. \end{aligned}$$

Thus $I_3 \to 0$ as $t \to \infty$. By a similar technique, we can prove that the rest of terms in (12) tend zero as $t \to \infty$. Therefore $(\phi x)(t) \to 0$ as $t \to \infty$. This completes the proof. □

Theorem 5. *Suppose that* $0 < B(t) < 1$. *If all solutions of (1) converge to zero, then (9) holds.*

Proof. Suppose that (9) does not holds. That is,

$$\lim_{t \to \infty} \prod_{s=t_0}^{t-1} (1 - B(s)) = \delta \neq 0. \quad (13)$$

So, from (13), we can write $\delta \neq 0$. Then, there exists a sequence $\{t_n\}$ approaching ∞, such that

$$\prod_{s=t_0}^{t_n-1}(1-B(s)) \to \delta \text{ as } n \to \infty.$$

For $x(t_0) \neq 0$, let x be a solution. Then,

$$\lim_{n\to\infty} |(x(t_0)+a(t_0-1)\,x(\tau(t_0)))| \prod_{s=t_0}^{t_n-1}(1-B(s)) = |(x(t_0)+a(t_0-1)\,x(\tau(t_0)))|\,\delta \neq 0. \tag{14}$$

From Lemma 1, $x(t_n)$ satisfies (4). On the other hand, we know that

$$\begin{aligned}
&\lim_{n\to\infty}\left[\sum_{r=t_0}^{t_n-1}\prod_{s=r+1}^{t_n-1}(1-B(s))\,|h(r)|\,|x(\tau(r))| - |a(t_n-1)|\,|x(\tau(t_n))|\right.\\
&+\sum_{r=t_0}^{t_n-1}\prod_{s=r+1}^{t_n-1}(1-B(s))\,|x(r+1)|\,|B(r)| + \sum_{r=t_0}^{t_n-1}\prod_{s=r+1}^{t_n-1}(1-B(s))\sum_{i=1}^{k}|b_i(r)|\,|x(\sigma_i(r))|\\
&\left.+\sum_{r=t_0}^{t_n-1}\prod_{s=r+1}^{t_n-1}(1-B(s))\sum_{j=1}^{l}\left|c_j(r)\right|\left|x(\tau_j(r))\right|\right] \; = \; 0.
\end{aligned} \tag{15}$$

Since all solutions tend zero, from (4), (14) and (15), we get

$$\lim_{n\to\infty} x(t_n) = |(x(t_0)+a(t_0-1)\,x(\tau(t_0)))|\,\delta \neq 0,$$

which contradicts all solutions of (1) converge to zero. The proof is completed. □

We end the paper with the following example.

Example 1. consider the mixed type neutral difference equation

$$\Delta x(t) + a(t)\,\Delta x(\tau(t)) + b_1(t)\,x(\sigma_1(t)) + c_1(t)\,x(\tau_1(t)) = 0, \tag{16}$$

with an assumed initial condition

$$x(t) = \psi(t) \text{ for } t \in [m(t_0), t_0] \cap \mathbb{Z},$$

where $t_0 = 0$, $m(t_0) = -2$, $\psi(t) = t/3$, $a(t) = \frac{1}{3^{t+2}}$, $b_1(t) = 1 - \frac{1}{2^t}$, $c_1(t) = \frac{1}{2^{t+1}}$, $\tau(t) = 3t/2$, $\sigma_1(t) = t/2 - 2$, $\tau_1(t) = 5t/2$. We have

$$B(t) = 1 - \frac{1}{2^{t+1}},\; \prod_{s=0}^{t-1}(1-B(s)) = \prod_{s=0}^{t-1}\frac{1}{2^{s+1}} \to 0 \text{ as } t \to \infty,$$

and

$$\begin{aligned}
&|a(t-1)| + \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\,|h(r)| + \sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\,|B(r)|\\
&+\sum_{r=t_0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\,|b_1(r)| + \sum_{r=0}^{t-1}\prod_{s=r+1}^{t-1}(1-B(s))\,|c_1(r)|\\
&= \frac{1}{3^{t+1}} + \sum_{r=0}^{t-1}\prod_{s=r+1}^{t-1}\frac{1}{2^{s+1}}\left|\frac{1}{3^{r+2}} - \frac{1}{3^{r+1}\times 2^{r+1}}\right| + \sum_{r=0}^{t-1}\prod_{s=r+1}^{t-1}\frac{1}{2^{s+1}}\left(1-\frac{1}{2^{r+1}}\right)\\
&+\sum_{r=0}^{t-1}\prod_{s=r+1}^{t-1}\frac{1}{2^{s+1}}\left(1-\frac{1}{2^{r}}\right) + \sum_{r=0}^{t-1}\prod_{s=r+1}^{t-1}\frac{1}{2^{s+1}}\times\frac{1}{2^{r+1}} \simeq 0.722 < 1.
\end{aligned}$$

Thus all the conditions of Theorem 4 are satisfied and every solution of (16) converges to zero.

3. Concluding remarks

In this article, a neutral mixed type difference equation is considered. The asymptotic behavior of solutions is obtained with a necessary and sufficient condition by using fixed point theorems. The results are supported with a suitable illustrative example.

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