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FLIP AND HOPF BIFURCATIONS OF DISCRETE-TIME FITZHUGH-NAGUMO MODEL

QAMAR DIN[1], SADAF KHALIQ

ABSTRACT. In this paper, dynamics of a two-dimensional Fitzhugh-Nagumo model is discussed. The discrete-time model is obtained with the implementation of forward Euler's scheme. We present the parametric conditions for local asymptotic stability of steady-states. It is shown that the two-dimensional discrete-time model undergoes period-doubling bifurcation and Neimark-Sacker bifurcation at its positive steady-state. Furthermore, in order to illustrate theoretical discussion some interesting numerical examples are presented.

Mathematics Subject Classification: 39A30, 40A05, 92D25, 92C50.
Key words and phrases: Fitzhugh-Nagumo model; Flip bifurcation; Hopf bifurcation.

1. Introduction

In 1961 FitzHugh and Nagumo [1] presented the following two-dimensional model:

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} c_1\left(x+y-\frac{x^3}{3}\right) \\ \frac{1}{c_1}\left(x-a_1+b_1y\right) \end{pmatrix}, \tag{1}$$

where a_1, b_1 and c_1 are positive constants. Using the forward Euler method to system (1), we get the discrete-time model as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+hc_1\left(x+y-\frac{x^3}{3}\right) \\ y-\frac{h}{c_1}\left(x-a_1+b_1y\right) \end{pmatrix}, \tag{2}$$

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[1] Corresponding Author

where $h>0$ is step size. For further biological relevance and dynamical analysis of some models that are very close to system (2), we refer to [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15], and references are therein. We investigate the existence of equilibria for (2) and local asymptotic stability of these steady-states by implementing linearized stability analysis techniques. Also, Neimark-Sacker bifurcation and period-doubling bifurcation are discussed.

2. Existence of equilibria and stability

The steady-states of (2) satisfy the following system of algebraic equations:

$$x = x + hc_1\left(x + y - \frac{x^3}{3}\right),\ y = y - \frac{h}{c_1}\left(x - a_1 + b_1 * y\right). \tag{3}$$

From (3), it is quite simple to obtain:

$$b_1 xs + 3(1-b_1)x - 3a_1 = 0,\ y = \frac{a_1 - x}{b_1}.$$

Next, we define the following quantity:

$$\Delta := 4b_1(b_1-1)^3 - 9a_1^2 b_1^2. \tag{4}$$

Then, it follows that

- If $\Delta > 0$, then system (2) has three distinct equilibrium points.
- If $\Delta = 0$, then system (2) has a multiple equilibrium points.
- If $\Delta < 0$, then system (2) has a unique positive equilibrium point.

For $a_1 = 4.92$ and $b_1 = 0.16$, we have $\Delta = -5.95649 < 0$ and existence for unique positive equilibrium is depicted in Figure 1. For $a_1 \in [0, 50]$ and $b_1 \in [0, 50]$, the region (blue) where $\Delta < 0$ and region (red) where $\Delta > 0$ are depicted in Figure 2. Mathematically, we have the following conditions for negativity and positivity of Δ:

- $\Delta < 0$ if and only if $0 < b_1 \leq 1$, or $b_1 > 1$ and $a_1 > \frac{2}{3}\sqrt{\frac{-1+3b_1-3b_1^2+b_1^3}{b_1}}$.
- $\Delta > 0$ if and only if $b_1 > 1$ and $a_1 < \frac{2}{3}\sqrt{\frac{-1+3b_1-3b_1^2+b_1^3}{b_1}}$.
- $\Delta = 0$ if and only if $b_1 > 1$ and $a_1 = \frac{2}{3}\sqrt{\frac{-1+3b_1-3b_1^2+b_1^3}{b_1}}$.

Now the Jacobian matrix of (2) evaluated at arbitrary equilibrium (x, y) is given by:

$$J(x,y) := \begin{pmatrix} 1 + \left(h - hx^2\right)c_1 & hc_1 \\ -\frac{h}{c_1} & 1 - \frac{hb_1}{c_1} \end{pmatrix}.$$

Moreover, the characteristic polynomial of $J(x, y)$ is given by:

$$\begin{aligned} P(\lambda) := {} & \lambda^2 - \left(2 - \frac{b_1 h}{c_1} + c_1 h\left(1 - x^2\right)\right)\lambda \\ & + 1 + h\left(c_1 + h - c_1 x^2\right) - \frac{b_1 h\left(1 + c_1 h\left(1 - x^2\right)\right)}{c_1}. \end{aligned} \tag{5}$$

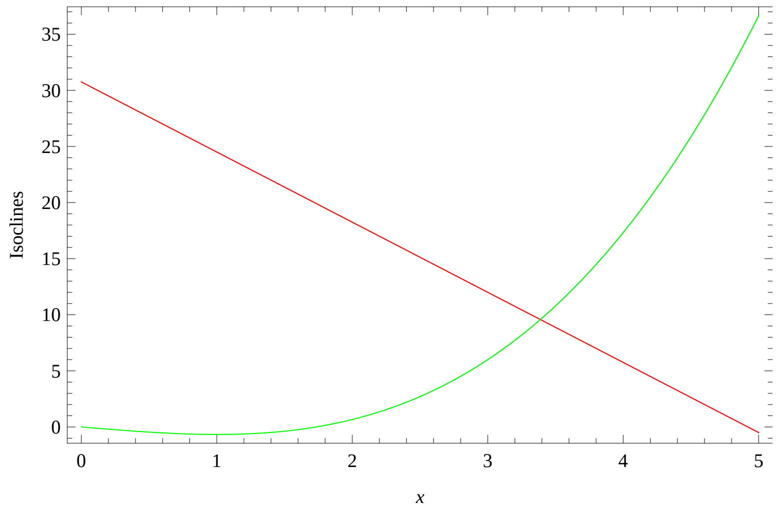

FIGURE 1. For $a_1 = 4.92$ and $b_1 = 0.16$ existence of unique positive equilibrium for (2)

Theorem 2.1. [16] *Assume that $\Delta < 0$, then unique positive equilibrium (x, y) has the following topological classification:*

(i) (x, y) is a sink if and only if

$$\left|2 - \frac{b_1 h}{c_1} + c_1 h\left(1 - x^2\right)\right| < 2 + h\left(c_1 + h - c_1 x^2\right) - \frac{b_1 h\left(1 + c_1 h\left(1 - x^2\right)\right)}{c_1} < 2.$$

(ii) (x, y) is a saddle if and only if

$$\left(2 - \frac{b_1 h}{c_1} + c_1 h\left(1 - x^2\right)\right)^2 - 4\left(1 + h\left(c_1 + h - c_1 x^2\right) - \frac{b_1 h\left(1 + c_1 h\left(1 - x^2\right)\right)}{c_1}\right) > 0,$$

and

$$\left|2 - \frac{b_1 h}{c_1} + c_1 h\left(1 - x^2\right)\right| > \left|2 + h\left(c_1 + h - c_1 x^2\right) - \frac{b_1 h\left(1 + c_1 h\left(1 - x^2\right)\right)}{c_1}\right|.$$

(iii) (x, y) is a source if and only if

$$\left|1 + h\left(c_1 + h - c_1 x^2\right) - \frac{b_1 h\left(1 + c_1 h\left(1 - x^2\right)\right)}{c_1}\right| > 1,$$

and

$$\left|2 - \frac{b_1 h}{c_1} + c_1 h\left(1 - x^2\right)\right| < \left|2 + h\left(c_1 + h - c_1 x^2\right) - \frac{b_1 h\left(1 + c_1 h\left(1 - x^2\right)\right)}{c_1}\right|.$$

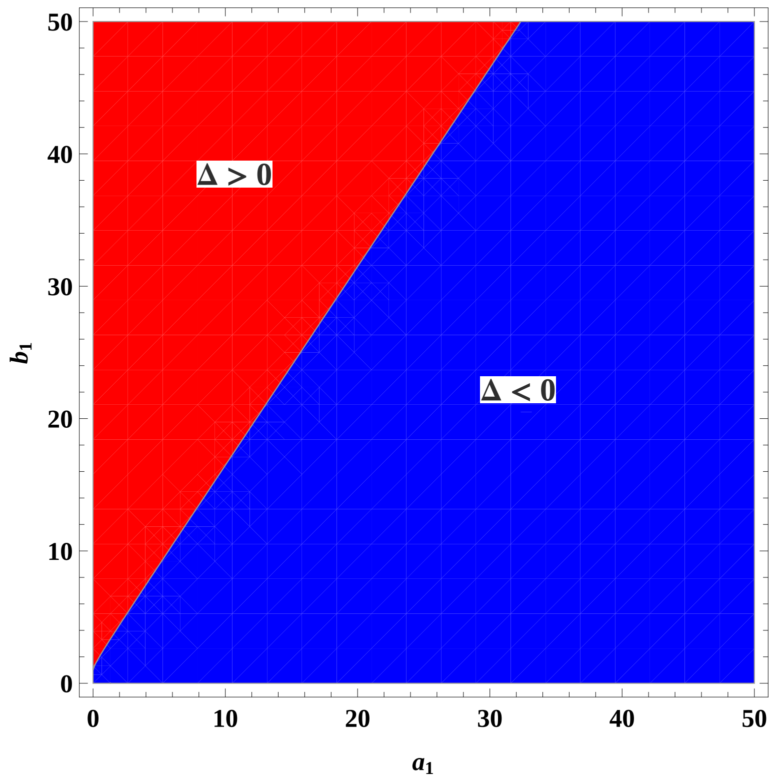

FIGURE 2. Regions for existence of various equilibria

(iv) (x, y) *is non-hyperbolic if and only if*

$$\left|2 - \frac{b_1 h}{c_1} + c_1 h\left(1 - x^2\right)\right| = \left|2 + h\left(c_1 + h - c_1 x^2\right) - \frac{b_1 h\left(1 + c_1 h\left(1 - x^2\right)\right)}{c_1}\right|,$$

or

$$\left(c_1 + h - c_1 x^2\right) - \frac{b_1\left(1 + c_1 h\left(1 - x^2\right)\right)}{c_1} = 0$$

and

$$\left|2 - \frac{b_1 h}{c_1} + c_1 h\left(1 - x^2\right)\right| \leq 2.$$

If we choose $b_1 = 0.45$, $c_1 = 0.15$, $h \in [0, 1]$ and $x \in [0, 10]$, then topological classification for unique positive point of system (2) is shown in Figure 3.

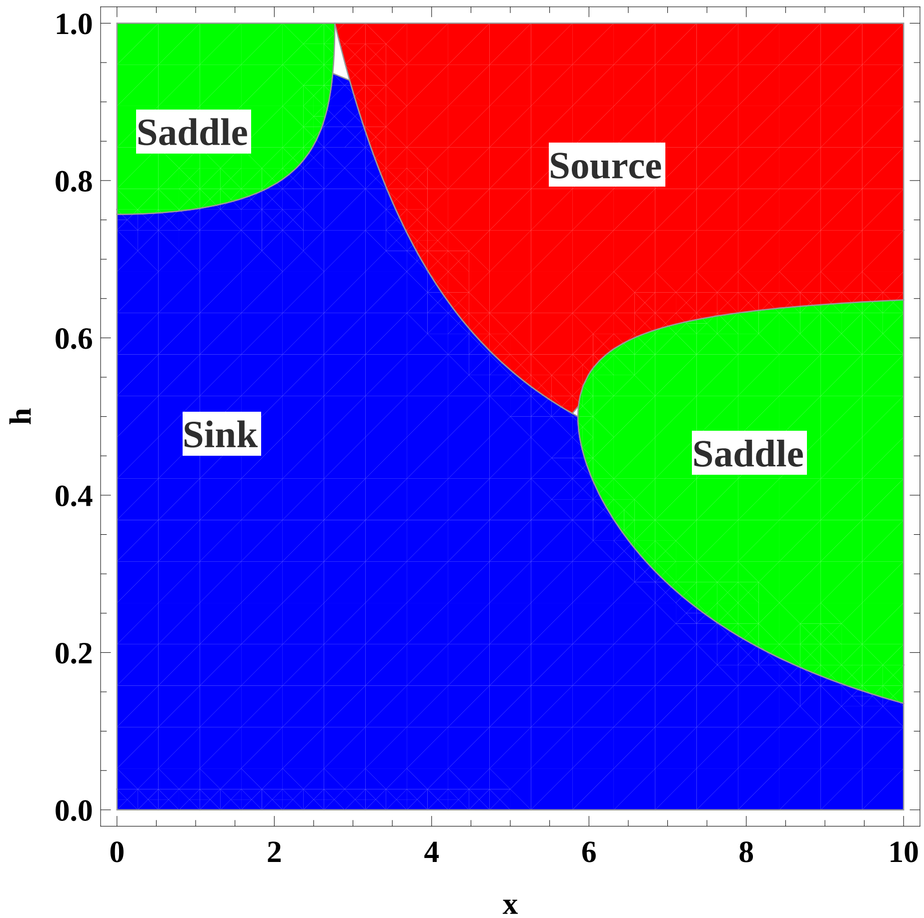

FIGURE 3. Topological classification for positive equilibrium

3. Bifurcation analysis

Studying bifurcation analysis for discrete-time models is a topic of great interest. Recently, there are many articles have published for the investigation for period-doubling and Neimark-Sacker bifurcations in discrete-time models [17, 18, 19, 20, 21, 22, 23]. In this section, we explore the parametric conditions under which system (2) undergoes period-doubling and Neimark-Sacker bifurcations at its unique positive equilibrium point. For this, first we discuss the emergence of period-doubling bifurcation at positive equilibrium of system (2). Assume that

$P(-1) = 0$, where $P(\lambda)$ is defined in (5), then system (2) undergoes period-doubling bifurcation as h varies in a small neighborhood of h_0 defined by

$$h_0 := \frac{b_1 + \left(-1 + x^2\right) c_1^2 - \sqrt{-4c_1^2 + \left(b_1 - \left(-1 + x^2\right) c_1^2\right)^2}}{\left(1 + \left(-1 + x^2\right) b_1\right) c_1},$$

or

$$h_0 := \frac{b_1 + \left(-1 + x^2\right) c_1^2 + \sqrt{-4c_1^2 + \left(b_1 - \left(-1 + x^2\right) c_1^2\right)^2}}{\left(1 + \left(-1 + x^2\right) b_1\right) c_1}.$$

Secondly, we assume that

$$\left(b_1 + \left(-1 + x^2\right) c_1^2\right)^2 \left(-4c_1^2 + \left(b_1 - \left(-1 + x^2\right) c_1^2\right)^2\right) < 0.$$

Then system (2) undergoes Neimark-Sacker bifurcation as parameter h varies in a small neighborhood of h_1 defined by:

$$h_1 := \frac{b_1 + \left(-1 + x^2\right) c_1^2}{\left(1 + \left(-1 + x^2\right) b_1\right) c_1}.$$

In order to verify aforementioned mathematical investigation for existence of period-doubling and Neimark-Sacker bifurcations, we choose particular parametric values for system (2) as follows:

Period-doubling bifurcation: Let $a_1 = 2.6$, $b_1 = 1.2$, $c_1 = 1.9$ and $h \in [0.3, 0.5]$. In this case, system (2) undergoes period-doubling bifurcation as h varies in a small neighborhood of $h_0 = 0.39$. Moreover, the bifurcation diagrams for period-doubling bifurcation are shown in Figure 4 and Figure 5. Moreover, maximum Lyapunov exponents (MLE) are shown in Figure 6 and a chaotic attractor is depicted in Figure 7.

Neimark-Sacker bifurcation: Taking $a_1 = 2.7$, $b_1 = 2.5$, $c_1 = 0.95$ and $h \in [0.65, 0.72]$. Then system (2) undergoes Neimark-Sacker bifurcation as h varies in a small neighborhood of $h_1 = 0.69$. The diagrams for Neimark-Sacker bifurcation are given in Figure 8 and Figure 9. Furthermore, MLE are shown in Figure 10 and phase portrait at $h = 0.69$ is depicted in Figure 11.

4. Conclusion

The qualitative behavior for a two-dimensional discrete-time Fitzhugh-Nagumo model is investigated. Euler's forward scheme is implemented to obtain the discrete counterpart of the continuous Fitzhugh-Nagumo model. It is investigated that discrete-time model has rich dynamical behavior as compare to its continuous counterpart. The topological classification for steady-state solutions is discussed. Furthermore, parametric conditions for the existence of period-doubling bifurcation and Neimark-Sacker bifurcation are analyzed by taking h as bifurcation parameter. At the end numerical simulations are provided to illustrate the theoretical discussion.

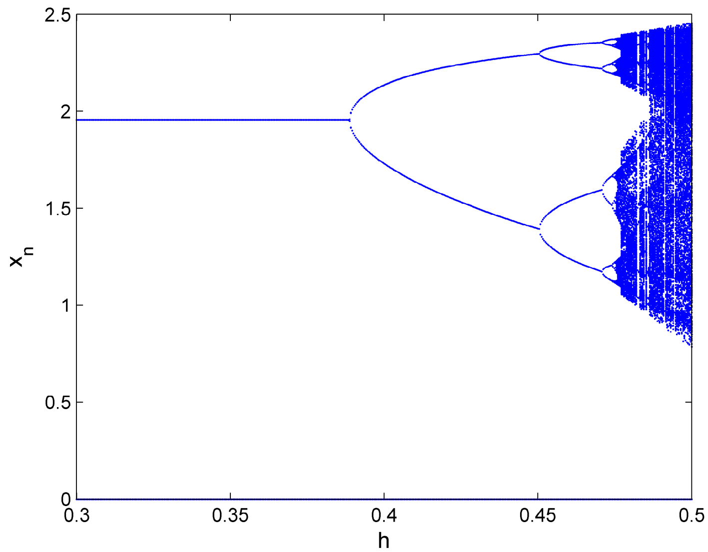

FIGURE 4. Bifurcation diagram for x_n

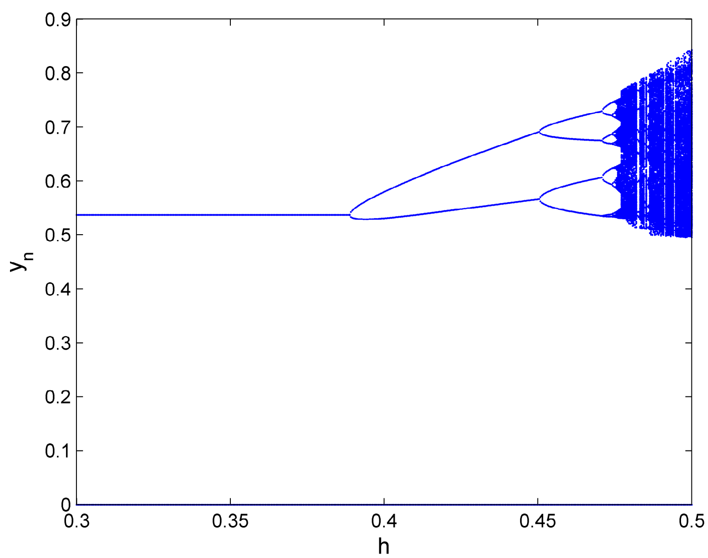

FIGURE 5. Bifurcation diagram for y_n

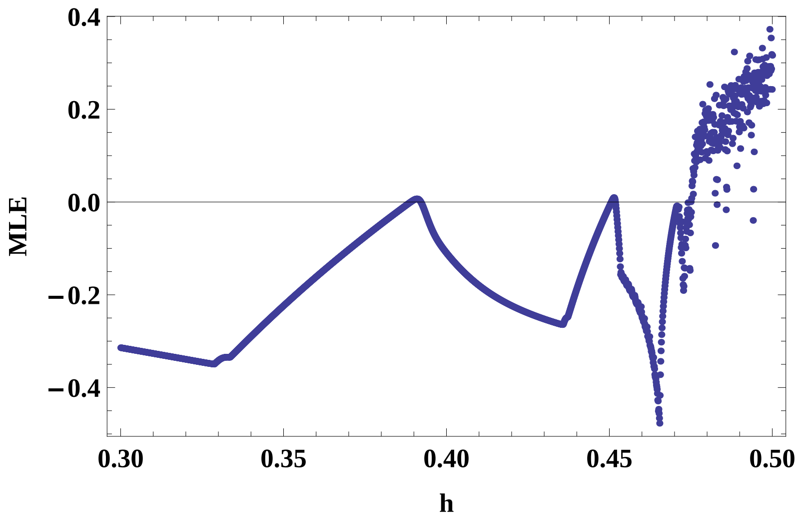

Figure 6. Maximum Lyapunov exponents

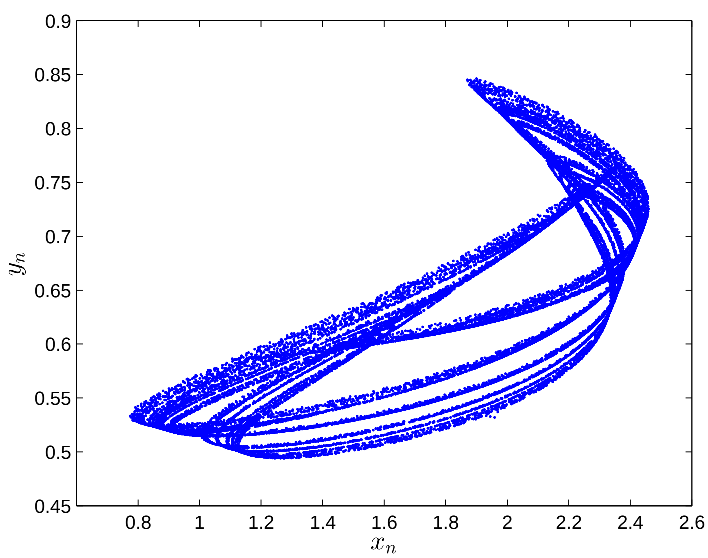

Figure 7. A chaotic attractor at $h = 0.5$

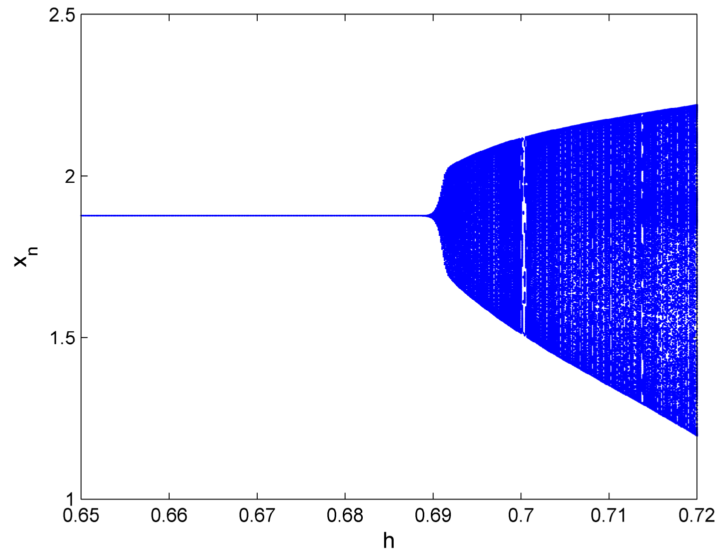

FIGURE 8. Bifurcation diagram for x_n

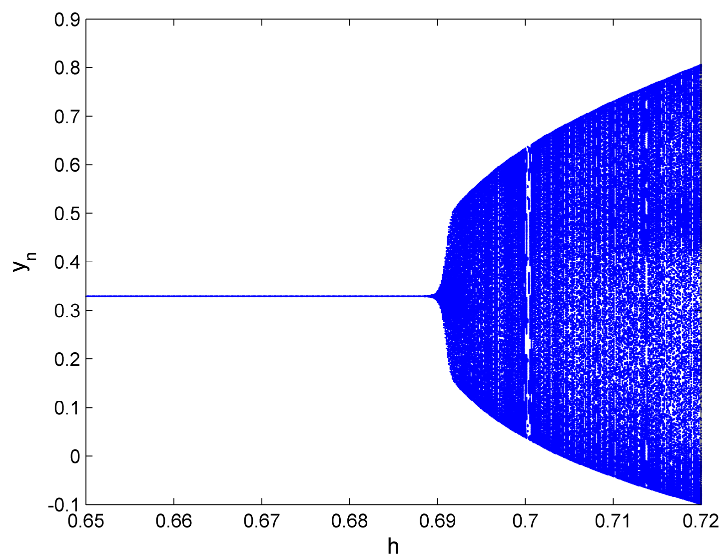

FIGURE 9. Bifurcation diagram for y_n

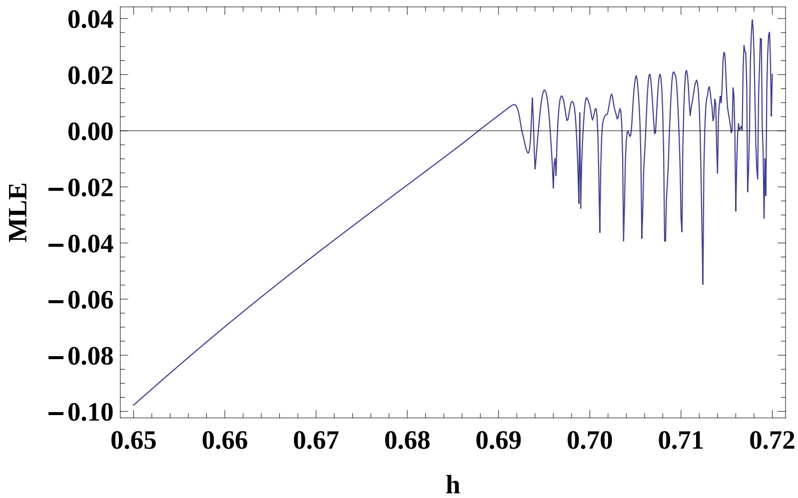

FIGURE 10. Maximum Lyapunov exponents

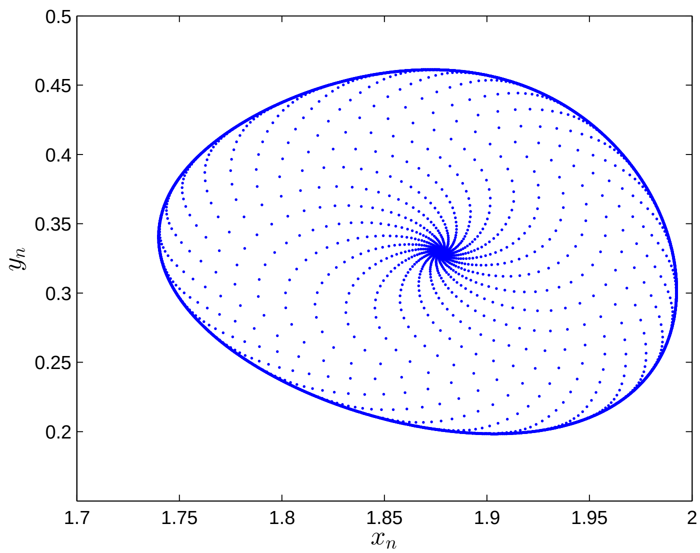

FIGURE 11. Phase portrait at $h = 0.69$

Competing Interests

The author(s) do not have any competing interests in the manuscript.

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Qamar Din
Department of Mathematics, The University of Poonch Rawalakot, Pakistan.
e-mail: qamar.sms@gmail.com

Sadaf Khaliq
Department of Mathematics, The University of Poonch Rawalakot, Pakistan.
e-mail: sadafkhaliq00@gmail.com