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THE WEIGHTED SQUARE INTEGRAL INEQUALITIES FOR SMOOTH AND WEAK SUBSOLUTION OF FOURTH ORDER LAPLACE EQUATION

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ABSTRACT. In this work we develop the weighted square integral estimates for the second derivatives of weak subsolution of forth order Laplace equation. It is natural generalization of inequalities develop for the Superharmonic functions in [1].

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Key words and phrases: fourth order Laplace equation; smooth subsolution; weak subsolution; adjoint operator.

1. Introduction

Among the most important of all partial differential equations are undoubtedly second order Laplace equation. Some of practical problems also covered by higher order Laplace equation. In [1], [2], [3], [4] and [5] the author develop the energy estimates for convex functions, 4-convex functions and also for super-harmonic functions. These estimates are very important in financial mathematics one can see [6], so it is also interesting to develop similar results for subsolution of fourth order Laplace equation. The fourth order Laplace equation with n variables is given as

$$\frac{\partial^4 u(x)}{\partial x_1^4} + \frac{\partial^4 u(x)}{\partial x_2^4} + \ldots + \frac{\partial^4 u(x)}{\partial x_n^4} = 0 \tag{1}$$

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Let us denote

$$\Delta^4 \equiv \frac{\partial^4}{\partial x_1^4} + \frac{\partial^4}{\partial x_2^4} + \ldots + \frac{\partial^4}{\partial x_n^4}$$

Then (1) becomes as

$$\Delta^4 u(x) = 0 \tag{2}$$

The function $u(x) \in C^4(D)$ is called subsolution (suppersolution) of fourth order Laplace equation (2) if

$$\Delta^4 u(x) \geq (\leq) 0 \tag{3}$$

The bounded measurable function $u(x)$ is called week subsolution of (1) if $u(x)$ satisfy

$$\int\limits_D u(x)\Delta^{*4}\psi(x)dx \geq 0 \tag{4}$$

It is trivial that Δ^4 is self-adjoint operator i.e.

$$\Delta^4 = \Delta^{*4}$$

Through out the paper we will use the following notations.

$$grad\ u(x) = \big(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, \ldots, \frac{\partial u(x)}{\partial x_n}\big)$$

$$grad^2\ u(x) = \big(\frac{\partial^2 u(x)}{\partial x_1^2}, \frac{\partial^2 u(x)}{\partial x_2^2}, \ldots, \frac{\partial^2 u(x)}{\partial x_n^2}\big)$$

We will organize the paper in the following way in second section we will derive the energy estimate for the fourth order Laplace equation. Also we approximate the week subsolution by the smooth ones. In the last section we will derived similar estimate regarding week subsolution.

2. The Weighted Energy Estimates for the Smooth Subsolution for the Fourth Order Laplace Equation.

Lemma 2.1. [5] *Assume $f(x) \in C^4(I)$ is the smooth and $h(x)$ is the smooth non-negative weight function having compact support. From the proof of Theorem 2.1 in* [5] *one can derive the identity in Lemma 2.1 as:*

$$\begin{aligned}\int\limits_I \left(f^{''}(x)\right)^2 h(x)dx &= \int\limits_I f(x)f^{(iv)}(x)h(x)dx - 2\int\limits_I f(x)f^{''}(x)h^{''}(x)dx \\ &+ \frac{1}{2}\int\limits_I f^2(x)h^{(iv)}(x)dx \end{aligned} \tag{5}$$

Theorem 2.2. *Let $u_i(x) \in C^4(D)$, $i = 1, 2$ be the smooth subsolutions of (2) over the domain $D \subseteq \mathbb{R}^n$ and also $\frac{\partial^2 u_i}{\partial x_j^2} \geq 0 \ \ \forall \ \ j = 1, 2, \ldots, n$, then we have the following energy estimate for the difference of the functions*

$$\begin{aligned}
&\int\limits_D \left|grad^2\big(u_2(x) - u_1(x)\big)\right|^2 h(x)dx \\
&\leq \int\limits_D \Bigg[\frac{1}{2}\big(u_2(x) - u_1(x)\big)^2 \\
&- \sup_{x\in D}\left|\big(u_2(x) - u_1(x)\big)\right|\big(u_2(x) + u_1(x)\big)\Bigg]\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}\,dx
\end{aligned} \tag{6}$$

where $h(x)$ is the non-negative smooth weight function satisfying

$$\begin{cases} h(x) = \frac{\partial h(x)}{\partial x_j} = \frac{\partial^2 h(x)}{\partial x_j^2} = \frac{\partial^3 h(x)}{\partial x_j^3} = 0 \quad j = 1, \ldots, n \ \ \forall \ x \in \partial D \\ \frac{\partial^2 h(x)}{\partial x_j^2} \leq 0 \quad j = 1, \ldots, n \quad \textit{for } x \in D. \end{cases} \tag{7}$$

Proof. Let

$$u(x) = u_2(x) - u_1(x) \tag{8}$$

Take

$$\begin{aligned}
\int\limits_D \left|grad^2 u(x)\right|^2 \ h(x)\ dx &= \int\limits_D \Bigg[\Big(\frac{\partial^2 u(x)}{\partial x_1^2}\Big)^2 + \Big(\frac{\partial^2 u(x)}{\partial x_2^2}\Big)^2 \\
&\quad + \ldots + \Big(\frac{\partial^2 u(x)}{\partial x_n^2}\Big)^2\Bigg] h(x)\,dx \\
&= \int\limits_D \Big(\frac{\partial^2 u(x)}{\partial x_1^2}\Big)^2 h(x)\,dx + \int\limits_D \Big(\frac{\partial^2 u(x)}{\partial x_2^2}\Big)^2 h(x)\,dx + \ldots + \\
&\quad \int\limits_D \Big(\frac{\partial^2 u(x)}{\partial x_n^2}\Big)^2 h(x)\,dx
\end{aligned} \tag{9}$$

By Lemma 2.1, the above may write as

$$\begin{aligned}
&= \sum_{j=1}^{n}\Big(\int\limits_D u(x)\frac{\partial^4 u(x)}{\partial x_j^4}\,h(x)dx - 2\int\limits_D u(x)\frac{\partial^2 u(x)}{\partial x_j^2}\,\frac{\partial^2 h(x)}{\partial x_j^2}\,dx + \frac{1}{2}\int\limits_D u^2(x)\,\frac{\partial^4 h(x)}{\partial x_j^4}dx\Big) \\
&\leq \int\limits_D |u(x)|\left|\sum_{j=1}^{n}\frac{\partial^4 u(x)}{\partial x_j^4}\right| h(x)dx + 2\int\limits_D |u(x)|\left|\sum_{j=1}^{n}\frac{\partial^2 u(x)}{\partial x_j^2}\ \frac{\partial^2 h(x)}{\partial x_j^2}\right| dx \\
&\quad + \frac{1}{2}\int\limits_D u^2(x)\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}\,dx
\end{aligned}$$

$$\leq \sup_{x\in D}|u(x)| \int\limits_D \left|\sum_{j=1}^{n} \frac{\partial^4 u(x)}{\partial x_j^4}\right| h(x)\,dx + 2\sup_{x\in D}|u(x)| \int\limits_D \left|\sum_{j=1}^{n} \frac{\partial^2 u(x)}{\partial x_j^2}\,\frac{\partial^2 h(x)}{\partial x_j^2}\right| dx$$

$$+\frac{1}{2}\int\limits_D u^2(x)\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}\,dx$$

Now replacing $u(x) = u_2(x) - u_1(x)$ we obtain,

$$\int\limits_D \left|grad^2\big(u_2(x)-u_1(x)\big)\right|^2\ h(x)\ dx$$

$$\leq \sup_{x\in D}\left|\big(u_2(x)-u_1(x)\big)\right| \int\limits_D \left|\sum_{j=1}^{n}\frac{\partial^4}{\partial x_j^4}\big(u_2(x)-u_1(x)\big)\right| h(x)\,dx$$

$$+2\sup_{x\in D}\left|\big(u_2(x)-u_1(x)\big)\right| \int\limits_D \left|\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}\big(u_2(x)-u_1(x)\big)\,\frac{\partial^2 h(x)}{\partial x_j^2}\right| dx$$

$$+\frac{1}{2}\int\limits_D \big(u_2(x)-u_1(x)\big)^2\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}\,dx$$

$$\leq \sup_{x\in D}|u_2(x)-u_1(x)|\sum_{j=1}^{n}\int\limits_D \Big(\frac{\partial^4 u_2(x)}{\partial x_j^4}+\frac{\partial^4 u_1(x)}{\partial x_j^4}\Big)h(x)\,dx$$

$$-2\sup_{x\in D}|u_2(x)-u_1(x)|\sum_{j=1}^{n}\int\limits_D \Big(\frac{\partial^2 u_2(x)}{\partial x_j^2}+\frac{\partial^2 u_1(x)}{\partial x_j^2}\Big)\frac{\partial^2 h(x)}{\partial x_j^2}\,dx$$

$$+\frac{1}{2}\int\limits_D \big(u_2(x)-u_1(x)\big)^2\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}\,dx \tag{10}$$

Again using the definition of weight function and integration by parts formula, we obtain

$$\int\limits_D \sum_{j=1}^{n}\frac{\partial^4 u_i(x)}{\partial x_j^4}\,h(x)\,dx = \int\limits_D u_i(x)\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}\,dx \qquad i=1,2$$

and

$$\int\limits_D \sum_{j=1}^{n}\frac{\partial^2 u_i(x)}{\partial x_j^2}\,\frac{\partial^2 h(x)}{\partial x_j^2}\,dx = \int\limits_D u_i(x)\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}\,dx \quad i=1,2$$

the above (10) becomes

$$\int\limits_D \left|grad^2\big(u_2(x)-u_1(x)\big)\right|^2 h(x)dx \leq$$

$$\int_D \Big(\frac{1}{2}\big(u_2(x)-u_1(x)\big)^2 - \sup_{x\in D}|u_2(x)-u_1(x)|\,\big(u_2(x)+u_1(x)\big)\Big)\sum_{j=1}^{n}\frac{\partial^4 h(x)}{\partial x_j^4}\,dx$$

□

Remark 2.1. Taking the supremum norm on above inequality we obtained

$$\begin{aligned}
&\int_D \left|grad^2\big(u_2(x)-u_1(x)\big)\right|^2 h(x)dx \\
&\leq \Big[\frac{1}{2}\left\|u_2(x)-u_1(x)\right\|_{L^\infty}^2 + \left\|u_2(x)-u_1(x)\right\|_{L^\infty} \\
&\times\big(\left\|u_1(x)\right\|_{L^\infty} + \left\|u_2(x)\right\|_{L^\infty}\big)\Big] \\
&\times \int_D \Delta^4 h(x)dx. \qquad (11)
\end{aligned}$$

Remark 2.2. The above estimates over domain $D \in \mathbb{R}^n$ also holds for arbitrary ball $B(x_o, r)$ with center x_o and radius r.

From now we use ball $B(x_o, r)$ as domain in $\mathbb{R}^n$.

3. The Weighted Energy Estimates for the Week Subsolution of Fourth Order Laplace Equation.

The continuous function $u(x)$ is said to be week subsolution of (2) if

$$\int_B u(x)\ \Delta^4\psi(x)\ dx \geq 0 \qquad (12)$$

where

$$\psi \in C_c^\infty(B).$$

Now we will approximate the week subsolution of (2) by the smooth ones. For this we will use the classical mollification. Define

$$\eta(x) = \begin{cases} c\ \ exp(\frac{1}{|x|^2-1}) & \text{if } |x|<1 \\ 0 & \text{if } |x| \geq 1 \end{cases} \qquad (13)$$

where $x \in \mathbb{R}^n$ and $c > 0$ is such that

$$\int_{\mathbb{R}^n} \eta(x)\ dx\ =\ 1. \qquad (14)$$

Now we define

$$u_\epsilon(x)\ =\ \epsilon^{-n}\int_B \eta\Big(\frac{x-y}{\epsilon}\Big)\ u(y)\,dy \qquad (15)$$

Let us denote

$$\eta_\epsilon(x-y)\ =\ \epsilon^{-n}\ \ \eta\Big(\frac{x-y}{\epsilon}\Big) \qquad (16)$$

By the definition $\eta_\epsilon(x-y)$, It is trivial that

$$\frac{\partial^4 \eta_\epsilon(x-y)}{\partial x_i^4} = \frac{\partial^4 \eta_\epsilon(x-y)}{\partial y_i^4} \qquad \forall\ i = 1, 2,\ \ .\ .\ .\ , n \tag{17}$$

so,

$$\Delta_x^4\ u_\epsilon(x) = \epsilon^{-n} \int\limits_B u(y)\ \Delta_y^4\ \eta_\epsilon(x-y)\ dy \tag{18}$$

where Δ_x^4 and Δ_y^4 are the fourth order Laplace operator w.r.t x and y respectively.
Let the ball $B_k = B(x_o, r_k)$ with $r_k = \frac{k+1}{k+2} r \quad , k = 1, 2, .$
The next theorem tells about the existence of smooth approximation.

Theorem 3.1. *Let $u(x)$ be the continuous week subsolution of* (2) *and convex over the ball B. Then for any $k = 1, 2, 3, \ldots,$ there exists $\hat{\epsilon} > 0$ such that $0 < \epsilon < \hat{\epsilon}$ each function $u_\epsilon(x)$ is smooth convex over the ball B_k and also*

$$\Delta^4 u_\epsilon(x) \geq 0 \ \ if \ \ x \in B_k.$$

Proof. For fixed $k = 1, 2, \ldots,$
Let

$$\hat{\epsilon} = \frac{r}{2(k+2)} \tag{19}$$

By the definition, it is trivial that $u_\epsilon(x)$ is infinitely differentiable and also from [7] $u_\epsilon(x)$ is smooth convex for each of its arguments.
we next claim that $\eta_\epsilon(x-y)$ has compact support in the ball B.
Take another ball $\hat{B}_k$ in the following way

$$\hat{B}_k = B(x_0, \frac{2k+3}{2k+4} r) \tag{20}$$

$$If \ \ y \notin \hat{B}_k \ \ then$$

$$|y-x| > \left| \frac{2k+3}{2k+4} - \frac{2k+2}{2k+4} \right| r = \frac{1}{2(k+2)}\, r > \epsilon \tag{21}$$

$$\Rightarrow \quad \eta_\epsilon(x-y) = 0 \tag{22}$$

Hence $\eta_\epsilon(x-y)$ has compact support.
Hence by the definition of week subsolution and also using (22), we get

$$\int\limits_B u(y)\, \Delta_y^4\, \eta_\epsilon(x-y)\, dy \ \geq 0. \tag{23}$$

$\square$

We will defined $h_k(x)$ as:

$$\begin{cases} h_k(x) > 0 \quad if \quad x \in B_k(x_0, r_k) \\ h_k(x) = 0 \quad if \quad x \in \partial B_k(x_0, r_k) \end{cases}$$

where $r_k = \frac{k+1}{k+2} r$

Theorem 3.2. *Let $u(x)$ be the continuous week subsolution of* (2) *and convex over the ball B then it possesses the following weak partial derivatives $\frac{\partial^2 u(x)}{\partial x_i^2}$, $i = 1, 2, \ldots, n$ over the ball B.*

Proof. For the existence of first derivative $\frac{\partial u(x)}{\partial x_i}$ $i = 1, 2, ..., n$ one can see [4]. Let us suppose the mollification $u_\epsilon(x)$ defined in (15) for the week subsolution of fourth order Laplace equation $u(x)$.
For the continuous function $u(x)$, the ball B, it is well-known fact that on compact set $\subseteq B$ we have the following uniform-convergence

$$\sup_k |u_\epsilon(x) - u(x)| \underset{\epsilon \to 0}{\longrightarrow} 0.$$

Let us denote $u_\epsilon(x)$ by $u_m(x)$ for $\epsilon = \frac{1}{m}$, $m = 1, 2, ...$ so above becomes

$$\sup_k |u_m(x) - u(x)| \underset{m \to \infty}{\longrightarrow} 0 \tag{24}$$

The ball B_k are compactly contained in the original ball B
From the theorem 3.1, we know that for any k=1, 2, ..., there exists m_k such that $u_m(x)$ is smooth subsolution of (2)
Take the ball B_{k+l} and write the inequality (11) for

$$u_1(x) = u_m(x) \ \ and \ \ u_2(x) = u_p(x)$$

$$\begin{aligned} &\int\limits_{B_{k+l}} \left| grad^2 u_p - grad^2 u_m \right|^2 \ h_{k+l} \ dx \\ &\leq \left[\frac{1}{2} \left\| u_p - u_m \right\|_{L^\infty}^2 + \left\| u_p - u_m \right\|_{L^\infty} \left(\left\| u_p \right\|_{L^\infty} + \left\| u_m \right\|_{L^\infty} \right) \right] \\ &\times \int\limits_{B_{k+l}} \left| \Delta^4 h_{k+l} \right| dx \end{aligned} \tag{25}$$

Let us denote

$$\alpha_{k+l} = \int\limits_{B_{k+l}} \left| \Delta^4 h_{k+l} \right| dx,$$

$$\hat{\alpha} = \inf_{x \in B_{k+l}} \ h_{k+l},$$

then (25) becomes

$$\hat{\alpha} \int\limits_{B_{k+l}} \left| grad^2 u_p - grad^2 u_m \right|^2 \ dx$$

$$\leq \alpha_{k+l}\left[\frac{1}{2}\left\|u_p-u_m\right\|^2_{L^\infty}+\left\|u_p-u_m\right\|_{L^\infty}\right.$$
$$\left.\times\left(\left\|u_p\right\|_{L^\infty}+\left\|u_m\right\|_{L^\infty}\right)\right] \tag{26}$$

where $\hat{\alpha}\neq 0$. Writing the left hand integral for the smaller ball B_k, we have

$$\hat{\alpha}\int\limits_{B_k}\left|grad^2u_p-grad^2u_m\right|^2\,dx$$
$$\leq \alpha_{k+l}\left[\frac{1}{2}\left\|u_p-u_m\right\|^2_{L^\infty}+\left\|u_p-u_m\right\|_{L^\infty}\right.$$
$$\left.\times\left(\left\|u_p\right\|_{L^\infty}+\left\|u_m\right\|_{L^\infty}\right)\right] \tag{27}$$

From (24), we have

$$\left\|u_p-u_m\right\|_{L^\infty_{(B_{k+l})}}\longrightarrow 0 \quad \text{as} \quad m,p\longrightarrow\infty$$

so (27) becomes.

$$\lim_{m,p\to\infty}\ \sum_{i=1}^{n}\int\limits_{B_k}\left(\frac{\partial^2u_p}{\partial x_i^2}-\frac{\partial^2u_m}{\partial x_i^2}\right)dx=0$$

The completeness of $L^2(B_k)$ ensure the convergence of above sequence. So there exist a class of measurable functions $v_{k,i}(x)\in L^2(B_k)$ such that

$$\sum_{i=1}^{n}\int\limits_{B_k}\left(\frac{\partial^2u_m}{\partial x_i^2}-v_{k,i}\right)^2dx\underset{m\to\infty}{\longrightarrow}0, k=1,2,...$$

we extend $v_{k,i}(x)$ trivially outside the ball B_k by 0.
Let us denote

$$v_i(x)=\lim_{k\to\infty}\sup_{x\in D}\ v_{k,i},\ \ i=1,2,...,n$$

It can be checked easily that

$$v_i(x)=v_{k,i}$$

for the ball B_k.
Next we claim that $v_i(x)$ represent the weak second order partial derivative $\frac{\partial^2u(x)}{\partial x_i^2}$ of $u(x)$.
Take $\psi(x)$ an arbitrary function having compact support in B.
Then suppose $\psi(x)\subset B_k$ from some k.
We have

$$\int\limits_{B}\frac{\partial^2u_m}{\partial x_i^2}\ \psi dx\ =\ \int\limits_{B}u_m\ \frac{\partial^2\psi}{\partial x_i^2}\ dx$$

for the integers $m \geq m_k$.
But we have the following convergence

$$|u_m(x) - u(x)| \underset{m\to\infty}{\longrightarrow} 0 \text{ on } B_k$$

and

$$\left\| \frac{\partial^2 u_m}{\partial x_i^2} - v_i(x) \right\|_{L^2(B_k)} \underset{m\to\infty}{\longrightarrow} 0$$

using these Limits, the above becomes.

$$\int\limits_{B_k} v_i(x)\ \psi(x)\ dx = \int\limits_{B_k} u(x) \frac{\partial^2 \psi}{\partial x_i^2}\ dx.$$

This shows that $v_i(x)$ $i = 1, 2, ..., n$ are the weak partial derivative of $u(x)$. Rewriting the inequality (11) for the functions $u_1(x) = 0$ and $u_2(x) = u_m(x)$ for $m \geq m_{k+l}$ over the ball B_{k+l}, we get

$$\int\limits_{B_{k+l}} \left|grad^2 u_m(x)\right|^2 h_{k+l}(x) dx \leq \frac{3}{2} \alpha_{k+l} \left\|u_m(x)\right\|^2_{L^\infty(B_{k+l})}$$

Taking limit $m \to \infty$, the above becomes

$$\int\limits_{B_{k+l}} \left|grad^2 u_m(x)\right|^2 h_{k+l}(x) dx \leq \frac{3}{2} \alpha_{k+l} \left\|u(x)\right\|^2_{L^\infty(B_{k+l})}$$

Now restricting the left hand side for the smaller ball B_k, we have

$$\int\limits_{B_k} \left|grad^2 u_m(x)\right|^2 h_{k+l}(x) dx \leq \frac{3}{2} \alpha_{k+l} \left\|u(x)\right\|^2_{L^\infty(B_{k+l})}$$

Now making limit as $m \to \infty$, we obtain

$$\int\limits_{B_k} \left|grad^2 u(x)\right|^2 h(x) dx \leq \frac{3}{2} \alpha_\infty \left\|u(x)\right\|^2_{L^\infty(B)} < \infty.$$

The left hand side of above increases as k increases and also bounded, so it will have finite limit, i.e

$$\int\limits_{B} \left|grad^2 u(x)\right|^2 h(x) dx \leq \frac{3}{2} \alpha_\infty \left\|u(x)\right\|^2_{L^\infty(B)} < \infty. \tag{28}$$

which completes the proof. □

Now we prove the inequality for the weak subsolution of fourth order Laplace equation.

Theorem 3.3. *Let* $u_i(x), i = 1, 2$ *be the continuous weak subsolution of* (2), *and also it satisfies*

$$\frac{\partial^2 u(x)}{\partial x_j^2} \geq 0 \qquad \forall j = 1, 2, \ldots, n$$

then the following is valid

$$\int\limits_B \left|grad^2 u_2(x) - grad^2 u_1(x)\right|^2 h(x)dx \leq \Bigg[\frac{1}{2}\|u_2(x)-u_1(x)\|^2_{L^\infty(B)} + \|(u_2(x)-u_1(x))\|_{L^\infty(B)} \times \big(\|u_1(x)\|_{L^\infty(B)} + \|u_2(x)\|_{L^\infty(B)}\big)\Bigg] \times \int\limits_B \left|\Delta^4 h(x)\right| dx \qquad (29)$$

where $h(x)$ is the weight function satisfying (7).

Proof. Take mollification $u_{m,i}(x)$, for $i = 1, 2$ of the continuous weak subsolution $u_i(x)$, for $i = 1, 2$ respectively.
Since for the ball B_{k+l}, there exists integer m_{k+l}, such that each function $u_{m,i}$, for $i = 1, 2$ is the smooth subsolution in the ball B_{k+l}, if $m > m_{k+l}$. Also we have the following convergence

$$\|u_{m,i}(x) - u_i(x)\|_{L^\infty(B_{k+l})} \underset{m\to\infty}{\longrightarrow} 0 \quad i = 1, 2.$$

Now writing the inequalities the function $u_{m,i}(x)$, i=1,2 and the ball B_{k+l} , we get

$$\int\limits_{B_{k+l}} \left|grad^2 u_{m,2}(x) - grad^2 u_{m,1}(x)\right|^2 h_{k+l}(x)dx$$
$$\leq \alpha_{k+l}\Bigg[\frac{1}{2}\|u_{m,2}(x)-u_{m,1}(x)\|^2_{L^\infty(B_{k+l})} + \|u_{m,2}(x)-u_{m,1}(x))\|_{L^\infty(B_{k+l})} \times \big(\|u_{m,1}(x)\|_{L^\infty(B_{k+l})} + \|u_{m,2}(x)\|_{L^\infty(B_{k+l})}\big)\Bigg] \qquad (30)$$

Taking Limit as $m \to \infty$, the latter inequality (30) becomes

$$\int\limits_{B_{k+l}} \left|grad^2 u_2(x) - grad^2 u_1(x)\right|^2 h_{k+l}(x)dx$$
$$\leq \alpha_{k+l}\Bigg[\frac{1}{2}\|u_2(x)-u_1(x)\|^2_{L^\infty(B_{k+l})} + \|u_2(x)-u_1(x)\|_{L^\infty(B_{k+l})} \times \big(\|u_1(x)\|_{L^\infty(B_{k+l})} + \|u_2(x)\|_{L^\infty(B_{k+l})}\big)\Bigg] \qquad (31)$$

Again writing the above inequality, the left hand side for smaller ball B_k and also taking Limit $l \to \infty$, we get, we finally obtain

$$\int\limits_{B_k} \left|grad^2 u_2(x) - grad^2 u_1(x)\right|^2 h(x)dx$$

$$\leq \alpha_\infty \left[\frac{1}{2} \|u_2(x) - u_1(x)\|^2_{L^\infty(B)} + \|u_2(x) - u_1(x)\|_{L^\infty(B)} \times \left(\|u_1(x)\|_{L^\infty(B)} + \|u_2(x)\|_{L^\infty(B)} \right) \right] \quad (32)$$

The above inequalities holds for all B_k, k=1,2,3,...,n. So also true for ball B. Which completes the proof. □

Conflict of Interests

The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

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