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OSCILLATORY BEHAVIOR OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS WITH A NON-POSITIVE NEUTRAL TERM

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Abstract. We shall present new oscillation criteria of second order nonlinear difference equations with a non-positive neutral term of the for $\Delta(a(t)(\Delta(x(t)-p(t)x(t-k)))^{\gamma})+q(t)x^{\beta}(t+1-m)=0$, with positive coefficients. Examples are given to illustrate the main results.

Mathematics Subject Classification: Write MSC.
Key words and phrases: Oscillation; Second order; Neutral difference equation; Non-positive neutral term.

1. Introduction

This paper deals with oscillatory behavior of all solutions of the nonlinear second order difference equations with a non-positive neutral term of the form

$$\Delta(a(t)(\Delta(x(t)-p(t)x(t-k)))^{\gamma})+q(t)x^{\beta}(t+1-m)=0. \qquad (1)$$

We assume that
(i) γ, β are the ratios of positive odd integers;
(ii) $\{a(t)\}$, $\{p(t)\}$ and $\{q(t)\}$ are positive real sequences for $t \geqslant t_0$, and $0 < p(t) < p_0 < 1$;
(iii) k is a positive integer and m is a nonnegative integer;
(iv) $h(t)=t-m+k+1 \leqslant t$, that is $m \geqslant k+1$.
We let

$$A(v,u)=\sum_{s=u}^{v-1}\frac{1}{a^{1/\gamma}(s)}, v \geqslant u \geqslant t_0,$$

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and assume that

$$A(t,t_0) \to \infty \text{ as } t \to \infty. \tag{2}$$

Let $\theta = \max\{k, m-1\}$. By a solution of equation (1), we mean a real sequence $\{x(t)\}$ defined for all $t \geqslant t_0 - \theta$ and satisfies equation (1) for all $t \geqslant t_0$. A solution of equation (1) is called oscillatory if its terms are neither eventually positive nor eventually negative, otherwise it is called non-oscillatory. If all solutions of the equation are oscillatory then the equation itself called oscillatory.

In recent years, there has been much research activity concerning the oscillation and asymptotic behavior of solutions of various classes of difference equations see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references cited therein. Meanwhile, there also have been numerous research for second order neutral functional difference equations, due to the comprehensive use in natural science and theoretical study. Some interesting recent results on the oscillatory and asymptotic behavior of second order difference equations can be found in [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. However, it seems that there are no known results regarding the oscillation of second order difference equations of type (1). More exactly existing literature does not provide any criteria which ensure oscillation of all solutions of equation (1). In view of the above motivation, our aim in this paper is to present sufficient conditions which ensure that all solutions of (1) are oscillatory.

2. Main results

For $t \geqslant T$ for some $T \geqslant t_0$ we let

$$\mu(t) = a^{1/\gamma}(t)A(t,T) \quad \textit{and} \quad Q(t) = \sum_{s=t}^{\infty} q(s).$$

We begin with the following new result.

Theorem 2.1. *Let conditions (i) - (iv) and equation (2) hold. If there exists a positive non-decreasing sequence $\{\rho(t)\}$ such that*

$$\limsup_{t\to\infty} \left(\rho(t)Q(t) + \sum_{s=t_2}^{t} \left[\rho(s)q(s) - \frac{\gamma^{\gamma}}{(1+\gamma)^{1+\gamma}} \frac{a(t-m+1)}{(\beta g(s))^{\gamma}} \left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^{\gamma}(s)} \right) \right] \right) = \infty, \tag{3}$$

where

$$g(t) = \begin{cases} 1, & \textit{when } \beta = \gamma, \\ c(A^{(\gamma-\beta)/\beta}(t)), & \textit{when } \beta > \gamma \textit{ for some constant } c > 0, \end{cases} \tag{4}$$

$$\limsup_{t\to\infty} \sum_{s=h(t)}^{t-1} A^{\beta}(h(t),h(s))q(s) > 1, \quad \textit{when } \beta = \gamma \tag{5}$$

and

$$\limsup_{t\to\infty} \sum_{s=h(t)}^{t-1} A^{\beta}(h(t),h(s))q(s) > c > 0, \quad \textit{when } \beta > \gamma, \tag{6}$$

then equation (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of equation (1), say $x(t) > 0$, $x(t-m+1) > 0$, $x(t-k) > 0$ for $t \geqslant t_1$ for some $t_1 \geqslant t_0$. It follows from equation (1) that

$$\Delta(a(t)(\Delta y(t))^{\gamma}) = -q(t)x^{\beta}(t-m+1), \tag{7}$$

where $y(t) = x(t) - p(t)x(t-k)$. Hence $a(t)(\Delta y(t))^{\gamma}$ is decreasing and of one sign. That is, there exists a $t_2 \geqslant t_1$ such that $\Delta y(t) > 0$ or $\Delta y(t) < 0$ for $t \geqslant t_2$. We claim that $\Delta y > 0$ for $t \geqslant t_2$. To prove it, we assume that $\Delta y(t) < 0$ for $t \geqslant t_2$. Then

$$a(t)(\Delta y(t))^{\gamma} \leqslant -c \quad for\ t \geqslant t_2,$$

where $c = -a(t_2)(\Delta y(t_2))^{\gamma} > 0$. Thus, we conclude that

$$y(t) \leqslant y(t_2) - c^{1/\gamma} \sum_{s=t_2}^{t-1} a^{-1/\gamma}(s).$$

By virtue of equation (2), $\lim_{t\to\infty} y(t) = -\infty$. Now, we consider the following two cases:

Case 1. If $x(t)$ is unbounded, then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ where $x(t_n) = \max\{x(s) : t_0 \leqslant s \leqslant t_n\}$. Since $t_n - k > t_0$ for all sufficiently large n,

$$x(t_n - k) = \max\{x(s) : t_0 \leqslant s \leqslant t_n - k\} \leqslant \max\{x(s) : t_0 \leqslant s \leqslant t_n\} = x(t_n).$$

Therefore, for all large n,

$$y(t_n) = x(t_n) - p(t_n)x(\tau(t_n)) \geqslant (1 - p(t_n))x(t_n) > 0,$$

where $\tau(t) = t - k$, which contradicts the fact that $\lim_{t\to\infty} y(t) = -\infty$.

Case 2. If $x(t)$ is bounded, then $y(t)$ is also bounded, which contradicts $\lim_{t\to\infty} y(t) = -\infty$. This completes the prove of the claim and conclude that $\Delta y(t) > 0$ for $t \geqslant t_2$.

Next, we have two cases to consider:

(I) $y(t) > 0$; (II) $y(t) < 0$, for $t \geqslant t_2$.

First assume that (I) holds. In view of equation (7) and $x(t) \geqslant y(t)$, we have

$$\Delta(a(t)(\Delta y(t))^{\gamma}) \leqslant -q(t)y^{\beta}(t-m+1) \leqslant 0. \tag{8}$$

It follows that

$$\begin{aligned} y(t) &= y(t_2) + \sum_{s=t_2}^{t-1} \frac{(a(s)(\Delta y(s))^{\gamma})^{1/\gamma}}{a^{1/\gamma}(s)} \\ &\geqslant a^{1/\gamma}(t)(\Delta y(t)) \sum_{s=t_2}^{t-1} a^{-1/\gamma}(s) \\ &:= \mu(t)\Delta y(t). \end{aligned} \tag{9}$$

Summing equation (8) from t to u, letting $u \to \infty$ and using the fact that $y(t)$ is increasing, we have

$$\begin{aligned} a(t)(\Delta y(t))^{\gamma} &\geqslant \sum_{s=t}^{\infty} q(s) y^{\beta}(s-m+1) \\ &\geqslant y^{\beta}(t-m+1)\left(\sum_{s=t}^{\infty} q(s)\right) \\ &:= Q(t) y^{\beta}(t-m+1). \end{aligned} \tag{10}$$

Suppose that $y(t) > 0$ for $t \geqslant t_2$. Define

$$w(t) = \rho(t)\frac{a(t)(\Delta y(t))^{\gamma}}{y^{\beta}(t-m)} \quad \text{for} \quad t \geqslant t_2. \tag{11}$$

Then, it follows that

$$w(t) = \rho(t)\frac{a(t)(\Delta y(t))^{\gamma}}{y^{\beta}(t-m)} \geqslant \rho(t)\left(\sum_{s=t}^{\infty} q(s)\right). \tag{12}$$

Now,

$$\begin{aligned} \Delta w(t) &= \Delta\left(\frac{\rho(t)}{y^{\beta}(t-m)}\right)(a(t+1)(\Delta y(t+1))^{\gamma}) + \Delta(a(t)(\Delta y(t))^{\gamma})\left(\frac{\rho(t)}{y^{\beta}(t-m)}\right) \\ &\leqslant -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right)w(t+1) - \left(\frac{\rho(t)}{\rho(t+1)}\right)\frac{\Delta y^{\beta}(t-m)}{y^{\beta}(t-m)}w(t+1). \end{aligned} \tag{13}$$

By the corollary of the Keller chain rule, for $0 < \beta \leq 1$, we have

$$\begin{aligned} \Delta y^{\beta}(t-m) &= \beta\int_0^1 \left[hy(g(t-m+1) + (1-h)y(t-m)\right]^{\beta-1}\Delta y(t-m)dh \\ &\geq \beta\int_0^1 \left[hy(t-m+1) + (1-h)y(t-m+1)\right]^{\beta-1}\Delta y(t-m)dh \\ &= \beta y^{\beta-1}(t-m+1)\Delta y(t-m), \quad 0 < \beta \leq 1, \end{aligned}$$

then using this in (13), we get

$$\begin{aligned} \Delta w(t) &\leqslant -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right)w(t+1) \\ &\quad - \beta\left(\frac{\rho(t)}{\rho(t+1)}\right)\frac{y^{\beta-1}(t-m+1)\Delta y(t-m)}{y^{\beta}(t-m)}w(t+1) \\ &\leqslant -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right)w(t+1) - \beta\left(\frac{\rho(t)}{\rho(t+1)}\right)\frac{\Delta y(t-m)}{y(t-m)}w(t+1). \end{aligned} \tag{14}$$

And for $\beta > 1$, we have

$$\begin{aligned}\Delta y^{\beta}(t-m) &= \beta \int_0^1 \left[hy(g(t-m+1) + (1-h)y(t-m)\right]^{\beta-1} \Delta y(t-m)dh \\ &\geq \beta \int_0^1 \left[hy(t-m) + (1-h)y(t-m)\right]^{\beta-1} \Delta y(t-m)dh \\ &= \beta y^{\beta-1}(t-m)\Delta y(t-m), \quad \beta > 1,\end{aligned}$$

then using this in equation (13), we get

$$\Delta w(t) \leqslant -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) - \beta\left(\frac{\rho(t)}{\rho(t+1)}\right)\frac{\Delta y(t-m)}{y(t-m)} w(t+1). \tag{15}$$

Thus, by equation (14) and equation (15), we obtain equation (15) hold for all $\beta > 0$. Since $(a(t)(\Delta y(t))^{\gamma})$ is decreasing, we have

$$\frac{\Delta y(t-m)}{\Delta y(t)} \geqslant \left(\frac{a(t)}{a(t-m)}\right)^{1/\gamma} \quad \text{and} \quad \frac{w(t+1)}{\rho(t+1)} \leqslant \frac{w(t)}{\rho(t)}. \tag{16}$$

Using equation (16) in equation (15), we obtain
$\Delta w(t) \leqslant -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) - \beta\left(\frac{\rho(t)}{\rho(t+1)}\right)\left(\frac{a(t)}{a(t-m)}\right)^{1/\gamma}\left(\frac{\Delta y(t)}{y(t-m)}\right) w(t+1)$.
Now,
$\frac{\Delta y(t)}{y^{\gamma/\beta}(t-m)} = \rho^{-1/\gamma}(t)a^{-1/\gamma}(t)w^{1/\gamma}(t) \geqslant \rho^{-1/\gamma}(t)a^{-1/\gamma}(t)\left(\frac{\rho(t)}{\rho(t+1)}\right)^{1/\gamma} w^{1/\gamma}(t+1).$

Thus,
$\Delta w(t) \leqslant -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) - \frac{\beta}{a^{1/\gamma}(t-m)}\left(\frac{\rho(t)}{\rho^{1+1/\gamma}(t+1)}\right) w^{1+(1/\gamma)}(t+1)y^{(\gamma-\beta)/\beta}(t-m)$,

and so,

$$\begin{aligned}\Delta w(t) \quad &\leqslant \quad -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) \\ &\quad -\frac{\beta\rho(t)}{a^{1/\gamma}(t-m)\rho^{1+1/\gamma}(t+1)} w^{1+1/\gamma}(t+1)y^{(\gamma-\beta)/\beta}(t-m).\end{aligned}$$

For the case $\beta = \gamma$, we see that $y^{(\gamma-\beta)/\beta}(t) = 1$ while for the case $\beta > \gamma$ and since $(a(t)(\Delta y(t)))^{\gamma}$ is decreasing, there exists a constant $c > 0$ such that

$$(a(t)(\Delta y(t)))^{\gamma} \leqslant c \quad \text{for} \quad t \geqslant t_2.$$

Summing this inequality from t_2 to $t-1$, we have

$$y(t) \leqslant y(t_2) + c^{1/\gamma}A(t, t_2),$$

and thus,

$$y^{(\gamma-\beta)/\beta}(t) \geqslant c^{(\gamma-\beta)/(\beta\gamma)}A^{(\gamma-\beta)/\beta}(t, t_2) := c^{*}A^{(\gamma-\beta)/\beta}(t, t_2),$$

where $c^* = c^{(\gamma-\beta)/(\beta\gamma)}$. Using those two cases and the definition of $g(t)$, we get

$$\begin{aligned}\Delta w(t) &\leqslant -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) \\ &\quad -\frac{\beta\rho(t)}{a^{1/\gamma}(t-m)\rho^{1+1/\gamma}(t+1)} g(t) w^{(1+\gamma)/\gamma}(t+1). \qquad (17)\end{aligned}$$

Setting

$$B := \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) \text{ and } C := \frac{\beta\rho(t)}{a^{1/\gamma}(t-m)\rho^{1+1/\gamma}(t+1)},$$

and using

$$Bu - Cu^{(1+\gamma)/\gamma} \leqslant \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}} \left(\frac{B^{\gamma+1}}{C^\gamma}\right),$$

(see [7]), we have

$$\Delta w(t) \leqslant -\rho(t)q(t) + \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}} \frac{a(t-m)}{(\beta g(t))^\gamma} \left(\frac{(\Delta\rho(t))^{\gamma+1}}{\rho^\gamma(t)}\right).$$

Summing this inequality from t_2 to $t-1$ we get

$$w(t) \leqslant w(t_2) - \sum_{s=t_2}^{t-1} \left[\rho(s)q(s) - \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}} \frac{a(s-m)}{(\beta g(s))^\gamma} \left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^\gamma(s)}\right)\right].$$

Taking into account the equation (12), we find

$$w(t_2) \geqslant \rho(t)Q(t) + \sum_{s=t_2}^{t-1} \left[\rho(s)q(s) - \frac{\gamma^\gamma}{(1+\gamma)^{\gamma+1}} \frac{a(s-m)}{(\beta g(s))^\gamma} \left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^\gamma(s)}\right)\right].$$

Taking the lim sup of both sides in the above inequality as $t \to \infty$, we obtain a contradiction to the equation(3).
Consider now case (II). If we put $z(t) = -y(t) > 0$ for $t \geqslant t_2$, then

$$z(t) = -y(t) = p(t)x(t-k) - x(t) \leqslant p(t)x(t-k),$$

or

$$x(t-k) \geqslant z(t) \text{ or } z(t) = x(t+k).$$

Using this inequality in equation (1), we have

$$\Delta(a(t)(\Delta z(t))^\gamma) \geqslant q(t)z^\beta(t-m+k+1) := q(t)z^\beta(h(t)). \qquad (18)$$

Clearly, we have $\Delta z(t) < 0$. Now, for $t_2 \leqslant u \leqslant v$, we may write

$$z(u) - z(v) = -\sum_{s=u}^{v} (a^{-1/\gamma}(s)(a(s)(\Delta z(s))^\gamma)^{1/\gamma}) \geqslant A(v,u)(-(a(v)(\Delta z(v))^\gamma)^{1/\gamma}),$$

for $t \geqslant s \geqslant t_2$, setting $u = h(s)$ and $v = h(t)$ in the above inequality we get

$$z(h(s)) \geqslant A(h(t), h(s)) \left(-(a(h(t))(\Delta z(h(t)))^\gamma)^{1/\gamma}\right).$$

Summing inequality (18) from $u = h(t) \geqslant t_2$ to $t-1$, we find

$$\begin{aligned} Z(t) &:= -a(h(t))(\Delta z(h(t)))^{\gamma} \\ &\geqslant (-a(h(t))(\Delta z(h(t)))^{\gamma})^{\frac{\beta}{\gamma}} \sum_{s=h(t)}^{t-1} A^{\beta}(h(t),h(s))q(s) \\ &= Z^{\beta/\gamma}(t) \sum_{s=h(t)}^{t-1} A^{\beta}(h(t),h(s))q(s), \end{aligned}$$

and hence

$$Z^{1-\beta/\gamma} \geqslant \sum_{s=h(t)}^{t-1} A^{\beta}(h(t),h(s))q(s).$$

Taking lim sup of both sides of this inequality as $t \to \infty$, we arrive at a contradiction to equation (5) when $\beta = \gamma$ and equation (6) when $\beta > \gamma$. This completes the proof. □

We note that Theorem 2.1 holds when $Q(t) < \infty$ and the additional term $\rho(t)Q(t)$ in equation (3) may improves some of the well-known existing results appeared in the literature. In the case when $Q(t)$ does not exists as $t \to \infty$, we see that equation (3) can be replaced by

$$\limsup_{t\to\infty} \sum_{s=t_2}^{t-1} \left[\rho(s)q(s) - \frac{\gamma^{\gamma}}{(1+\gamma)^{\gamma+1}} \frac{a(t-m+1)}{(\beta g(s))^{\gamma}} \left(\frac{(\Delta\rho(s))^{\gamma+1}}{\rho^{\gamma}(s)}\right)\right] = \infty, \quad (19)$$

and the conclusion of Theorem 2.1 holds.

For the non-neutral equations, that is, equation (1) when $p(t) = 0$ and $q(t)$ is either non-negative or non-positive for all large t, equation (1) is reduced to the equation

$$\Delta\left(a(t)(\Delta x(t))^{\gamma}\right) + \delta q(t) x^{\beta}(t+1-m) = 0,$$

where $\delta = \pm 1$. From Theorem 2.1, we extract the following immediate results.

Corollary 2.2. *Let conditions* (i)-(iii) *and equation (2) hold. If there exists a positive function* $\rho(t)$ *and* $\Delta\rho(t) \geqslant 0$ *such that equation (3) holds, then equation* $(1,+1)$ *is oscillatory.*

Proof. The proof is contained in the proof of Theorem 2.1-Case (I) and hence is omitted. □

We note that Corollary 2.2 is related to some of the results in [4, 5, 6, 12, 13, 14, 15, 16, 17] and the references cited therein.

Corollary 2.3. *Let conditions* (i)-(iv) *and equation (2) hold. If equation (5) or (6) holds, then every bounded solution of equation* $(1,-1)$ *is oscillatory.*

Proof. The proof is contained in the proof of Theorem 2.1-Case (II) and hence is omitted. □

The following examples are illustrative.

Example 2.4. Consider the neutral equation

$$\Delta^2\left(x(t)-\frac{1}{2}x(t-3)\right)+8x(t-7)=0. \tag{20}$$

Here, $k=3$ and $m=8$ and so, $h(t)=t-3$. All conditions of Theorem 2.1 with equation (3) be replaced by equation (19) are satisfied and hence equation (20) is oscillatory.

Next, we present the following interesting results.

Theorem 2.5. *Let the hypotheses of Theorem 2.1 hold with $\Delta\rho \leqslant 0$ for $t \geqslant t_0$ and equation (3) be replaced by*

$$\limsup_{t\to\infty}\left[\rho(t)Q(t)+\sum_{s=t_0}^{t-1}\rho(s)q(s)\right]=\infty. \tag{21}$$

Then equation (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of equation (1), say $x(t)>0$, $x(t-k)>0$, $x(t-m+1)>0$ for $t \geqslant t_1$. Proceeding as in the proof of Theorem 2.1, we conclude that $\Delta y(t)>0$ for $t\geqslant t_2$ and we have two cases to consider: (I) $y(t)>0$ or $y(t)<0$ for $t\geqslant t_2$.
Case (I). Suppose that $y(t)>0$. As in the proof of Theorem 2.1, we obtain (16). Thus,

$$\Delta w(t)\leqslant -\rho(t)q(t).$$

Summing this inequality and using equation (10) we arrived at the desired contradiction. □

Example 2.6. Consider the neutral equation

$$\Delta^2\left(x(t)-\frac{1}{2}x(t-1)\right)+x(t-1)=0. \tag{22}$$

Here, $k=1$ and $m=1$ and so, $\rho(t)=t$. All conditions of Theorem 2.1 with equation (3) be replaced by equation (21) are satisfied and hence equation (22) is oscillatory.

In the following theorem we employ different approaches to replace equation (3) in Theorem 2.1.

Theorem 2.7. *Let the hypotheses of Theorem 2.1 hold with $\gamma\leqslant 1$, and equation (3) be replaced by*

$$\limsup_{t\to\infty}\left[\rho(t)Q(t)+\sum_{s=t_0}^{t-1}\rho(s)q(s)-\frac{a^{1/\gamma}(s-m+1)(\Delta\rho(s))^2}{4\beta g(s)\rho(s)Q^{1/\gamma-1}(s+1)}\right]=\infty. \tag{23}$$

Then equation (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of equation (1), say $x(t) > 0$, $x(t-k) > 0$, $x(t-m+1) > 0$ for $t \geqslant t_1$. Proceeding as in the proof of Theorem 2.1, we conclude that $\Delta y(t)$ for $t \geqslant t_2$ and $y(t)$ satisfies either (I) or (II) for $t \geqslant t_2$. If (I) holds, then as in the proof of Theorem 2.1, we obtain (17) and using (12) we get

$$\begin{aligned} \Delta w(t) &\leqslant -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) \\ &- \frac{\beta\rho(t)}{a^{1/\gamma}(t-m+1)\rho^{1+1/\gamma}(t+1)} g(t) w^{1+1/\gamma}(t+1) \\ &\leqslant -\rho(t)q(t) + \left(\frac{\Delta\rho(t)}{\rho(t+1)}\right) w(t+1) \\ &- \frac{\beta\rho(t)}{a^{1/\gamma}(t-m+1)\rho^{2}(t+1)} g(t) Q^{1/\gamma-1}(t+1) w^{2}(t+1) \\ &= -\rho(t)q(t) - \Big(\sqrt{\frac{\beta\rho(t)}{a^{1/\gamma}(t-m+1)\rho^{2}(t+1)} g(t) Q^{(1/\gamma)-1}(t+1)}\, w(t+1) \\ &- \frac{\frac{\Delta\rho(t)}{\rho(t+1)}}{2\sqrt{\frac{\beta\rho(t)}{a^{1/\gamma}(t-m+1)\rho^{2}(t+1)} g(t) Q^{(1/\gamma)-1}(t+1)}}\Big)^2 + \frac{a^{1/\gamma}(t-m+1)(\Delta\rho(t))^2}{4\beta g(t)\rho(t) Q^{1/\gamma-1}(t+1)} \\ &\leqslant -\rho(t)q(t) + \frac{a^{1/\gamma}(t-m+1)(\Delta\rho(t))^2}{4\beta g(t)\rho(t) Q^{(1/\gamma)-1}(t+1)}. \end{aligned}$$

The rest of the proof is similar to that of Theorem 2.1 and hence is omitted. □

Example 2.8. Consider the neutral equation

$$\Delta^2\left(x(t) - \frac{1}{3}x(t-2)\right) + x(t-3) = 0. \tag{24}$$

Here, $k = 2$ and $m = 4$ and so, $\gamma = 1$, $\rho(t) = t$. All conditions of Theorem 2.1 with equation (3) be replaced by equation (23) are satisfied and hence equation (24) is oscillatory.

Next, we present the following new and easily verifiable oscillation criteria for equation (1).

Theorem 2.9. *Let conditions* (*i*)-(*iv*) *and equation (2) hold. Assume that equation (5) and*

$$\limsup_{t\to\infty} A^{\beta}(t-m+1, t_0) Q(t) > 1 \tag{25}$$

hold when $\beta = \gamma$, *and equation (6) and*

$$\limsup_{t\to\infty} A^{\beta}(t-m+1, t_0) Q(t) > 0 \tag{26}$$

hold when $\beta < \gamma$, *then equation (1) is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of equation (1), say $x(t) > 0$, $x(t-k) > 0$, $x(t-m+1) > 0$ for $t \geqslant t_1$ for some $t_1 \geqslant t_0$. Proceeding as in the proof of Theorem 2.1, we conclude that $\Delta y(t) > 0$ for $t \geqslant t_2$ and $y(t)$ satisfies either (I) or (II) for $t \geqslant t_2$. If (I) holds, then as in the proof of Theorem 2.1, we obtain

(9) and (10). Using the facts that $\sigma(t) \leqslant t$ is decreasing, we find

$$w(t) := a(t)(\Delta y(t))^{\gamma} \geqslant Q(t)\mu^{\beta}(\tau(t))(\Delta y(t-m+1))^{\beta}$$

$$= Q(t)\mu^{\beta}(t-m+1)(a^{-\beta/\gamma}(t-m+1))(a(t-m+1)(\Delta y(t-m+1))^{\gamma})^{\beta/\gamma}$$

$$\geqslant Q(t)\mu^{\beta}(t-m+1)(a^{-\beta/\gamma}(t-m+1))(a(t)(\Delta y(t))^{\gamma})^{\beta/\gamma}$$

$$= Q(t)\mu^{\beta}(t-m+1)(a^{-\beta/\gamma}(t-m+1))w^{\beta/\gamma}(t),$$

or

$$\begin{aligned} w^{1-\beta/\gamma}(t) \quad &\geqslant Q(t)\mu^{\beta}(t-m+1)(a^{-\beta/\gamma}(\tau(t)) \\ &= Q(t)\left(\sum_{s=t_2}^{t-m+1} a^{-1/\gamma}(s)\right)^{\beta} = A^{\beta}(t-m+1,t_2)Q(t). \end{aligned}$$

Taking $\limsup$ of both sides of this inequality as $t \to \infty$, we arrive at a contradiction to equation (25) when $\beta = \gamma$ and equation (26) when $\beta < \gamma$. The proof of case (II) is similar to that of Theorem 2.1 and hence is omitted. □

Example 2.10. Consider the neutral equation

$$\Delta\left(\Delta\left(x(t) - \frac{1}{3}x(t-2)\right)\right)^2 + x(t-3) = 0. \tag{27}$$

Here, $k = 2$ and $m = 4$ and so, $\gamma = 2$, $\beta = 1$. Equation (26) of Theorem 2.5 are satisfied and hence equation (27) is oscillatory.

For equation (1) with advanced argument, we present the following result.

Theorem 2.11. *Let $\tau(t) \geqslant t$, conditions (i)-(iii) and equation (2) hold. Assume that the conditions*

$$\limsup_{t\to\infty} A(t,t_0)Q^{1/\gamma}(t) > 1, \tag{28}$$

,

$$\limsup_{t\to\infty} \sum_{u=h(t)}^{t-1}\left(\frac{1}{a(u)}\sum_{s=u}^{t} q(s)\right)^{1/\gamma} > 1 \tag{29}$$

hold when $\gamma = \beta$ and the conditions

$$\limsup_{t\to\infty} A(t,t_0)Q^{1/\gamma}(t) = \infty, \tag{30}$$

$$\limsup_{t\to\infty} \sum_{u=h(t)}^{t-1}\left(\frac{1}{a(u)}\sum_{s=u}^{t} q(s)\right)^{1/\gamma} > 0 \tag{31}$$

hold when $\beta < \gamma$, then equation (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of equation (1), say $x(t) > 0$, $x(t-k) > 0$, $x(t-m+1) > 0$ for $t \geqslant t_1$ for some $t_1 \geqslant t_0$. Proceeding as in the proof of Theorem 2.1 and consider the two cases (I) and (II). First, suppose case (I) holds. From equation (10), we have

$$(\Delta y(t))^{\gamma} \geqslant \left(\frac{Q(t)}{a(t)}\right) y^{\beta}(t-m+1),$$

or

$$\Delta y(t) \geqslant \left(\frac{Q(t)}{a(t)}\right)^{1/\gamma} y^{\beta/\gamma}(t-m+1).$$

Using above inequality in (9), we get

$$\begin{aligned} y(t) &\geqslant \mu(t)\Delta y(t) \\ &\geqslant \mu(t)\left(\frac{1}{a(t)}\sum_{s=t}^{\infty} q(s)\right)^{1/\gamma} y^{\beta/\gamma}(t-m+1) \\ &\geqslant A(t,t_2)Q^{1/\gamma}(t)y^{\beta/\gamma}(t), \end{aligned}$$

or

$$y^{1-\beta/\gamma}(t) \geqslant A(t,t_2)Q^{1/\gamma}(t).$$

Taking lim sup of both sides of this inequality as $t \to \infty$, we arrive at a contradiction to equation (28) when $\beta = \gamma$ and equation (30) when $\beta < \gamma$. If (II) holds, then as in the proof of Theorem 2.1-Case (II), we obtain equation (18). Summing this inequality from u to $t-1$,

$$(a(t)(\Delta z(t)))^{\gamma} - (a(u)(\Delta z(u)))^{\gamma} \geqslant \sum_{s=u}^{t} q(s)z^{\beta}(h(s))$$

or

$$-\Delta z(u) \geqslant \left(\frac{1}{a(u)}\sum_{s=u}^{t} q(s)z^{\beta}(h(s))\right)^{1/\gamma} \geqslant \left(\frac{1}{a(u)}\sum_{s=u}^{t} q(s)\right)^{1/\gamma} z^{\beta/\gamma}(h(t)).$$

Summing this inequality from $h(t) \geqslant t_2$ to $t-1$, we arrive at a contradiction to equation (29) when $\beta = \gamma$ or equation (31) when $\beta < \gamma$. □

Example 2.12. Consider the neutral equation

$$\Delta\left(\Delta\left(x(t) - \frac{1}{3}x(t-2)\right)\right)^2 + 2x(t-3) = 0. \tag{32}$$

Here, $k = 2$ and $m = 4$ and so, $\gamma = 2$, $\beta = 1$. Condition (30) and (31) of Theorem 2.5 are satisfied and hence equation (32) is oscillatory.

We may note that corollaries similar to Corollaries 2.2 and 2.3 can be also drawn from Theorems 2.5 and 2.7. The details are left to the reader.

3. Conclusion

We present seven sufficient conditions which ensure that all solutions of (1) are oscillatory. The corresponding examples are given to illustrate the significance of the results. From this, the oscillation criteria for the n order equation are similar.

Competing Interests

The author(s) do not have any competing interests in the manuscript.

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