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Investigating the Imperfection of the B – S Model: A Case Study of an Emerging Stock Market

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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ABSTRACT

The Black – Scholes (B-S) model is one of the widely used models in the pricing of financial option. The B-S model like most other models hinges on assumptions; one of which is the normality condition. A lot of researches have shown that using the log-return of developed market index that this assumption does not hold. We have shown in this paper using the log return from 1st January 2010 to 31st December 2012 in an emerging (Nigerian Stock Exchange) market All Share Index (ASI) to further support the reports of the non - normality condition of the B-S model.

Keywords: Scholes; Log – return; all share – index; financial options.

1. INTRODUCTION

The financial market is the hub of any economy. Basically, in developing economy, the major financial market is the stock market. In developed economy, the major transactions found in the stock market are more than just trading in the stocks of quoted companies; their major transactions are found in financial derivatives. Financial derivative is a contract whose value is determine by the value of one or more underlying assets. The main group of underlying assets is stocks, foreign currencies, interest rates, index etc. There are four main types of derivatives namely options, futures, forwards and swaps.

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An index, tracks the value of a basket of stocks. FTSE100, S&P500, Don Jones Industrial, NASDAQ Composite BEL20, are examples of some well known indexes. In fact, nearly all stock markets worldwide have their own index. In the Nigerian Stock Exchange (NSE), they have the All Share Index (ASI).

An Option gives the holder the right but not the obligation to buy or sell an asset in the future at a price that is agreed upon today. Every option has the exercise date or expiration date, exercise price or strike price and command a premium, also called the price of the option. The option is said to be exercised when the holder chooses to buy or sell the underlying stock. The writer of the option is the other party to the contract. The holder (writer) is said to be in the (short) position of the option contract. We have the following types of options – European option, American option, Exotic option.

In trading in an option, there are different models in use. One of the commonest and widely used models is the Black – Scholes (B-S) model. But one should however know that the B-S model hinges on several assumptions, which includes that there were no market frictions, like taxes and transaction cost. However, empirical evidences suggest that the classical B-S model does not describe some statistical properties of financial time series. Our focus in this paper will be on the non – normality of the log return of stocks. To test this, we use the log return of the ASI of the NSE. Our aim in this paper is to project the imperfection of the B-S model using the ASI of the NSE as a case study. The reminder of this paper is divided into sections as follows: Section 2.1 deals with some selected literature review, section 3.1 mathematical formulations and section 4.1, conclusion.

2.1 Literature Review

The Nigerian Stock Exchange (NSE) started business in 1961 with 19 securities. According to Olowe [1], as at 1998, there were 264 securities listed on the NSE, made up of 186 equity securities listed on the NSE, and 78 debt securities. By 2006, the number has increased to 288 securities, made up of 202 equity securities and 86 debt securities. The Nigerian Stock Exchange ASI has also grown overtime. The ASI grew from134.6 on January 1986 to 65,005.48 by March 18, 2008. There was a drastic fall of ASI due to the worldwide economic meltdown of 2008/2009. As at January 16, 2009, the Nigerian Stock Exchange ASI stood at 27,108.54. There has been a steady rise of the index recently, the ASI currently (as at May, 2013) stand at 32,000 plus points.

Black and Scholes [2], developed a model to price a European call option written on non dividend paying stock. Rubinstein [3] states that B-S option pricing model is the most widely used formula, with embedded probabilities, in human history.

Empirical investigations concede that the B-S model produces bias in its estimation. The assumptions of a historical instantaneous volatility measure and an underlying lognormal distribution do not hold. Yang [4], suggests implied volatility approach. Duan [5], suggests the tail properties of the underlying lognormal distributions are too small. Black and Scholes [2], using S&P500 option index data 1966-1969 suggest the variance that applies over the option produces a price between the model price and market price. In Black and Scholes [2], it was proposed that evidence volatility is not stationary. Galai [6], confirms Black and Scholes [2], assumptions that historical instantaneous volatility need be relaxed. Beckers [7] tested the B-S assuming that the historical instantaneous volatility of the underlying stock is a function of the stock price, using S&P500 index option 1972 -1977, and Beckers [7], finds the underlying stock in an inverse function of the stock price. Jackwerth [8], show the distribution of the S&P500 before 1987 exert lognormal distributions, but have since deteriorated to resemble leptokurtosis and negative skewness. Several studies seek to increase the tail properties of the lognormal distribution by incorporating a jump-diffusion process or stochastic volatility. Trautmann and Beinert [9], estimated parameters of a jump-diffusion process on German Capital Market, against B-S model. They found out that option prices generated through a jump-diffusion model are not comparable from those obtained from B-S model. Win [10], analysed the empirical evidence that suggest that classical B-S model does not describe the statistical properties of financial time series very well. There focus was on two main problems; that the log-return does not behave according to a normal distribution and that it was observed that the volatilities changes stochastically over time and are clustered.

In this paper, we probed further, with empirical evidence that B-S model do not describe the statistical properties of financial time series adequately with focus on, the log-return of the Nigerian Stock Exchange ASI not being a normal distribution.

3. MATHEMATICAL FORMULATIONS

Let a stock price follow

$$dS = \mu S dt + \sigma S dW, \tag{1}$$

where μ is the trend, σ the volatility and W follows a standard Brownian motion. Now suppose that f is the price of a call option. The variable f must be some function of S and t, hence by Ito's lemma, we have

$$dF(t,s) = \frac{\partial F}{\partial S} dS + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{ds^2}(ds^2)$$

$$dF = \frac{\partial F}{\partial S}(\mu s dt + \sigma s dW) + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{ds^2}(\mu s dt + \sigma s dW)^2$$

$$dF = \frac{\partial F}{\partial S}(\mu s dt + \sigma s dW) + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{ds^2}\sigma^2 s^2 dt)$$

$$dF = \frac{\partial F}{\partial S}\mu s dt + \frac{\partial F}{\partial S}\sigma s dW + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{ds^2} + \sigma^2 s^2 dt.$$

$$dF = \left[\frac{\partial f}{\partial S}\mu s + \frac{\partial f}{\partial t} + \frac{1}{2}s^2\sigma^2\frac{\partial^{2f}}{\partial S^2}\right]\partial t + \frac{\partial f}{\partial s}\sigma s dW. \tag{2}$$

The Brownian motion underlying f and s are the same and can be eliminated by choosing the appropriate portfolio of the stock and the derivative. Here, we choose a portfolio of -1: derivative

$$+\frac{\partial f}{\partial s}: shares$$

The holder is short one derivative and the long one amount $\frac{\partial f}{\partial s}$ of shares. We define Θ as the value of the portfolio and we have

$$\Theta = -f + \frac{\partial f}{\partial s} s. \tag{3}$$

The change $\partial\Theta$ in time interval ∂t of the value of the portfolio is given by

$$\partial\Theta = -\partial f + \frac{\partial f}{\partial s} \partial s. \tag{4}$$

Substituting (2) into (4), we have

$$\partial\Theta = \left[-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial s^2} \sigma^2 \ s^2 \right] \partial t. \tag{5}$$

The portfolio is now risk-free due to the elimination of ∂W term. And must now earn a return similar to other terms risk –free securities. Therefore

$$\partial\Theta = r\Theta \partial t, \tag{6}$$

where r is the risk-free interest rate. Substituting (3) and (5) into (6) we have,

$$\left[\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial s^2} \sigma^2 s^2 \right] \partial t = r \left[f - \frac{\partial f}{\partial s} s \right] \partial t, \tag{7}$$

Thus we have,

$$\frac{\partial f}{\partial t} + rs \frac{\partial f}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} = rf. \tag{8}$$

This is the black-Scholes-Merton differential equation.

For a non-dividend paying stock, early exercise is never optimal and the price of an American call carries the same value as that of the European. The respective lower and upper boundary conditions are given by

$$\begin{aligned} &C(S,t.K), C(S,t,K) \geq S_t - ke^{-r(T-t)} \\ &C(S,t.K), C(S,t,K) \leq S_t. \end{aligned} \tag{9}$$

The pay-off condition in the lower and the upper boundary conditions are given in (9), and they should be satisfied by the P.D.E.

The Payoff condition is $f(S, t = T) = \max(S - K, 0)$.The lower and upper boundary conditions are given by (9).

So let $r = T - t$, where T is the expiration time and t is the present time, then (8) can be written as

$$\frac{\partial f}{\partial r} = \frac{\delta^2}{2}.s^2\frac{\partial^2 f}{\partial s^2} + rs\frac{\partial f}{\partial s} \quad rf. \tag{10}$$

Taking $y = \ln S$, we have

$$\frac{\partial f}{\partial s} = \frac{1}{s}\frac{\partial f}{\partial y},$$
$$\frac{\partial^2 f}{\partial s^2} = -\frac{1}{s}\frac{\partial f}{\partial y} + \frac{1}{s^2}\frac{\partial^2 f}{\partial y^2}. \tag{11}$$

Introducing $w(y,T) = e^{rT} f(y,T)$, using (11), the Black-Scholes equation becomes a diffusion equation,

$$\frac{\partial W}{\partial T} = -\frac{\sigma^2}{2}\frac{\partial^2 W}{\partial y^2} + \left[r - \frac{\sigma^2}{2}\right]\frac{\partial W}{\partial y}. \tag{12}$$

And has a fundamental solution as a normal function on

$$\phi(y,T) = \frac{1}{\delta\sqrt{2\lambda T}}\exp\left[-\frac{\left[y + (r - \frac{\partial^q}{2})T)^2\right]}{2\partial^2 T}T\right]. \tag{13}$$

The solution to (12) is

$$W(y,T) = \int_{-\infty}^{\infty} w(\xi,0)\phi(y-\xi,T)d\xi. \tag{14}$$

Using the payoff conditions and the fundamental solutions obtained in (13), we obtain

$$w(y,T) = \frac{1}{\partial\sqrt{2\pi T}}\int_{-\infty}^{\infty}\max(e^{\xi} - k,0)\exp\left[-\frac{\left[y - \xi + (r - \frac{\partial^2}{2})T\right]^2 d\xi}{2\partial^2 T}\right],$$

$$= \frac{1}{\partial\sqrt{2\pi T}}\int_{InK}^{\infty}\max(e^{\xi} - k)\exp\left[\frac{[-y - \xi + r - \frac{\partial^2}{2}T]^2}{2\partial^2 T}\right]. \tag{15}$$

We denote the distribution function for a normal variable by $N(x)$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{U^2}{2}du} \tag{16}$$

we can express (15) as

$$w(y,T)=\frac{1}{\partial\sqrt{2\pi T}}\int_{InK}^{\infty}e^{\xi}\exp\left[-\frac{(-\xi+A^2)}{2\partial^2 T}\right]d\xi, \tag{17}$$

$$=\frac{-K}{\partial\sqrt{2\pi T}}\int_{InK}^{\infty}\exp\left[\frac{-(-\xi+A^2)}{2\partial^2 T}\right]d\xi,$$

where $A=y+(r-\frac{\partial^2}{2}T=\ln S+\frac{\partial^2}{2}T).$

Considering the second term in the RHS of (17). Let

$$z=(\frac{-\xi-A}{\partial\sqrt{T}}). \tag{18}$$

Using (18), $d\xi$ becomes

$$d\xi=-\partial\sqrt{T}dz. \tag{19}$$

And the limits of (17) are

$$Z=-\infty\, when\, \xi=\infty.$$

$$Z=\frac{InK+A}{\partial\sqrt{T}}=\frac{-\ln K+\ln S+(r-\frac{\partial^2}{2}T)}{\partial\sqrt{T}}\equiv d_2\ when\ \xi=\ln K. \tag{20}$$

Changing the variable from $\xi\ to\, Z$ the second term of (17) becomes

$$\frac{k}{\sqrt{2\pi}}\int_{d_2}^{-\infty}e^{-\frac{Z^2}{2}}dZ=\frac{k}{\sqrt{2\pi}}\int_{-\infty}^{d_2}e^{-\frac{Z^2}{2}}dZ=-KN(d_2). \tag{21}$$

The integrand of the first term in (17) is expressed as

$$e^{\xi e}\exp\left[\frac{-(\xi+A)^2}{2\partial^2 T}\right],$$

$$=\exp\left[\frac{-\xi^2-2(A-\delta^2 T)\xi+(A+\partial^2 T)\xi+A^2}{2\partial^2 T}\right],$$

$$=\exp\left[\frac{-\xi^2-2(A-\partial^2 T)\xi+(A+\partial^2 T)\xi+(A+\partial^2 T)^2-(A+\partial^2 T)^2+A^2}{2\partial^2 T}\right],$$

$$=\exp\left[\frac{-(\xi-(a+\partial^2 T))^2}{2\partial^2 T}\right]. \tag{22}$$

Using the definition of A, we have

$$e^{\frac{1}{2}\delta^2 T+A}=e^{y+rT}=Se^{rT}. \tag{23}$$

Inserting (21) and (22) into the first term of (23), it becomes

$$\frac{1}{\partial\sqrt{2\pi T}}Se^{rT}\int_{InK}^{d_1}\exp\left[\frac{-(\xi-(A+\partial^2 T))^2}{2\partial^2 T}\right]d\xi. \tag{24}$$

By changing the variables again we have

$$\frac{1}{\sqrt{2\pi}}Se^{rT}\int_{InK}^{d_1}e^{-\frac{Z^2}{2}}dz = Se^{rT}N(d_1). \tag{25}$$

The last line of (25) can be rewritten as

$$w(y,T) = e^{rT}SN\left[\frac{\ln(\frac{S}{K})+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right] - KN\left[\frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right] \tag{26}$$

And it implies that

$$C = SN(d_1) - Ke^{-rt}Nd_2\,, \tag{27}$$

$$where\ d_1 = \frac{\ln(\frac{S}{K})+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\ and\ d_2 = d_1 - \sigma. \tag{28}$$

This is the B-S formula for the price at time zero of a European call option on a non dividend paying stock.

Although the model has caused a major breakthrough in the pricing of stock options and has had huge influence on the way trader's price and hedge options, the normality assumption is questionable and this is what this paper wants to probe further using the ASI of the NSE.

Let X be a random variable, then

$$E(X) = \mu \quad and \quad Var(X) = E[X-(E(X)^2)].$$

A normal distribution is said to be symmetric when its skewness is zero and a kurtosis of three.

Skewness is given as

$$\frac{\sum_{i=1}^{N}\left(X_i-\overline{X}\right)^3}{NS^3},$$

and kurtosis is given as

$$\frac{\sum_{i=1}^{N}\left(X_i-\overline{X}\right)^4}{NS^4},$$

where X_i = random sample, $\overline{X}$ = sample mean, N = sample size and S = standard deviation.

We considered the daily returns over the period 1st January, 2010 to 31st December, 2012 of the ASI of the Nigerian stock exchange.

4. RESULTS AND FINDINGS

Skeweness is generally used to measure whether a distribution is asymmetric. For a normal/symmetric distribution, the skeweness is zero. If a distribution has a longer tail to the left than to the right, it is said to have a negative skeweness. But if the reverse is true, it is said to have a positive skewness.

The table below shows the Skeweness of some of the popular indices calculated with the Nigerian All Share Index (NASI) included.

Table 1, shows the skewness of some popular indexes with that of the Nigerian ASI highlighted in red as 0.1081, this is more than zero.

Table 1. Skewness of some indices

Index	Skeweness
SP500	-0.4423
NASDAQ-COMPOSITE	-0.5474
NIKKEI-225	0.1187
AEX	-0.3132
DAX	-0.4329
SSMI	-0.3595
BEL-20	-0.4141
NASI	0.1081

Source: Authors' field data (2013)

Since for a normal distribution the skewness is zero, looking at the empirical data, above we can see that there is significant skewness.

5. KURTOSIS

This is defined as the degree, to which a distribution is skewed. For the Normal distribution (mesokurtic), the kurtosis is 3. If the distribution has a flatter top (platykurtic), the kurtosis is less than 3. If the distribution has a high peak (leptokurtic), the kurtosis is greater than 3.

Table 2, below depicts the kurtosis some of the popular indices in the world calculated with the Nigerian All Share Index (NASI) included.

Table 2. Kurtosis of Some Indices

Index	Kurtosis
SP500	6.96
NASDAQ-COMPOSITE	5.81
NIKKEI-225	4.76
AEX	5.10
DAX	4.67
SSMI	5.40
BEL-20	5.40
NASI	7.98

Source: Authors' field data (2013)

From the Table 2 above, the ASI of the Nigerian stock exchange highlighted in red is 7.98, this is clearly above 3.

The above data in Table 2 shows that the value for kurtosis clearly bigger than 3, indicating that the tails of the normal distribution go much faster to zero than the empirical data suggests and that the distribution is much more peaked.

Since, the value in Table1 is not zero and that of Table 2 is greater than 3, the two conditions for normality are not met. This clearly brings to bear that one of the main assumptions of the B-S model is not met.

5. CONCLUSIONS

We have shown in this work, using the log- return from January 2010 to December 2012 on the All Share Index (ASI) of the Nigerian stock market that the B-S model normality assumption does not completely hold; this was done using the conditions for normality, skewness and kurtosis. Our findings further buttress that of other researchers done in developed markets as shown in Tables 1 and 2 of section 4.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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