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Weak Implication on Monadic Heyting Algebras

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Abstract

In this paper, we introduce an implication operation, called *weak implication*, which will be quite useful in order to characterize subdirectly irreducible monadic Heyting algebras. Furthermore, it is shown that deductively semisimple algebras are the non trivial ones such that the subalgebra of constants is a Tarski algebra with first element, i.e. a Boolean algebra, as it is mentioned by A. Monteiro and O. Varsavsky in 1957 (*Algebras de Heyting monádicas*, Actas de las X Jornadas de la Unión Matemática Argentina, Bahía Blanca, (1957), (52–62). Finally, it is stated that some of the results established for monadic Heyting algebras are also valid for monadic generalized Heyting algebras.

Keywords: Bounded distributive lattices; heyting algebras; monadic heyting algebras.

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1 Preliminaries

We refer the reader to the bibliography listed here as [1, 2, 3, 4] for specific details of the many basic notions and results of Heyting algebras including distributive lattices and universal algebras considered in this paper.

In 1995, A. Monteiro described Heyting algebras, which he called Brouwer algebras, as algebras $\langle L, \wedge, \vee, \rightarrow, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$ satisfying the identities:

(H1) $x \rightarrow x = 1$,

(H2) $(x \rightarrow y) \wedge y = y$,

(H3) $x \wedge (x \rightarrow y) = x \wedge y$,

(H4) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,

(H5) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,

(H6) $0 \wedge x = 0$.

It is simple to verify that in any Heyting algebra L the following properties hold ([3]) :

(H7) $x \wedge y \leq z$ if and only if $y \leq x \rightarrow z$,

(H8) $x \leq y$ if and only if $x \rightarrow y = 1$,

(H9) $y \leq x \rightarrow y$,

(H10) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$,

(H11) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$,

(H12) $x \rightarrow (y \rightarrow z) = (x \wedge y) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

In 1957, A. Monteiro and O. Varsavsky ([5]) (see also [6, 7, 8]) considered a generalization of monadic Boolean algebras ([9]) and defined monadic Heyting algebras as follows:

A monadic Heyting algebra (or $MH-$algebra) is a triple $(L, \nabla, \triangle)$ where L is a Heyting algebra and ∇, $\triangle$ are unary operations on L which satisfy the following conditions:

(E1) $\nabla 0 = 0$,

(E2) $x \leq \nabla x$,

(E3) $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$,

(E4) $\nabla(x \vee y) = \nabla x \vee \nabla y$,

(E5) $\nabla\nabla x = \nabla x$,

(E6) $\triangle 1 = 1$,

(E7) $\triangle x \leq x$,

(E8) $\triangle(x \wedge y) = \triangle x \wedge \triangle y$,

(E9) $\triangle\triangle x = \triangle x$,

(E10) $\triangle\nabla x = \nabla x$,

(E11) $\nabla\triangle x = \triangle x$.

From the above definition, it immediately follows that ∇ is a quantifier on L ([10, p.185]) and $\triangle$ is the dual of an additive closure operator ([3, p.47]), called existential quantifier and universal quantifier respectively.

In what follows we will denote by $\boldsymbol{MH}$ the variety of $MH-$algebras and they will usually be indicated by L.

The well-known results announced here for $\boldsymbol{MH}$ will be used throughout the paper.

(MH1) If $L \in \boldsymbol{MH}$, then it is simple to verify that $x = \triangle x$ if, and only if, $x = \nabla x$ and that $\triangle(L)$ is a subalgebra of L.

(MH2) Let $L \in \boldsymbol{MH}$. Then, $D \subseteq L$ is a monadic deductive system if it verifies: (D1) $1 \in D$, (D2) $x, x \to y \in D$ imply $y \in D$ and (D3) $x \in D$ implies $\triangle x \in D$. We will denote by $\mathcal{D}(L)$ the sets of all monadic deductive systems of L.

(MH3) Let $L \in \boldsymbol{MH}$ with more than one element and let $Con(L)$ be the lattice of all congruences on L. Then, $Con(L) = \{R(D) : D \in \mathcal{D}(L)\}$, where $R(D) = \{(x, y) \in L \times L : x \to y, y \to x \in D\}$. Besides, the lattices $Con(L)$ and $\mathcal{D}(L)$ are isomorphic considering the mappings $\theta \longmapsto [1]_\theta$ and $D \longmapsto R(D)$, which are mutually inverse, where $[x]_\theta$ stands for the equivalence class of x modulo θ.

2 Subdirectly Irreducible MH-Algebras

Next, our attention is focused on characterizing subdirectly irreducible as well as simple $MH-$algebras.

Lemma 2.1. *Let $L \in \boldsymbol{MH}$ and $a \in \triangle(L)$. Then, $[a) = \{x \in L : a \leq x\}$ is the monadic deductive system generated by a.*

Proof. It is routine. □

Theorem 2.2. *Let $L \in \boldsymbol{MH}$ with more than one element. Then, the following conditions are equivalent:*

(i) *L is subdirectly irreducible,*

(ii) *$\triangle(L) \setminus \{1\}$ has last element.*

Proof. (i) ⇒ (ii): By the hypothesis and (MH3), there is $D_0 \in \mathcal{D}(L) \setminus \{1\}$ such that $D_0 \subseteq D$ for all $D \in \mathcal{D}(L) \setminus \{1\}$. Since $D_0 \neq \{1\}$, there is $p \in D_0$, $p \neq 1$. From $\triangle p \neq 1$ and $\triangle p \in D_0$ it follows that $[\triangle p) \subseteq D_0$ and, consequently by Lemma 2.1, we have that $[\triangle p) = D_0$. On the other hand, let $x \in \triangle L \setminus \{1\}$. Hence, Lemma 2.1 allows us to assert that $[x) \in \mathcal{D}(L) \setminus \{1\}$ and so, we have that $[\triangle p) \subseteq [x)$. Therefore, $\triangle p$ is the last element of $\triangle L \setminus \{1\}$.
(ii) ⇒ (i): Let $D \in \mathcal{D}(L) \setminus \{1\}$. Then, there is $x \in D$, $x \neq 1$ and so, $\triangle x \in \triangle(L) \setminus \{1\}$. If p is the last element of $\triangle(L) \setminus \{1\}$, hence $\triangle x \leq p$. Therefore, $D_0 = [p) \subseteq D$ for all $D \in \mathcal{D}(L) \setminus \{1\}$ and by well-known results of universal algebra ([4]) we conclude the proof. □

Definition 2.1. Let L be a Heyting algebra. A $uq-$pair on L is a pair $(\nabla, \triangle)$ of unary operations on L such that $(L, \nabla, \triangle) \in \boldsymbol{MH}$.

Definition 2.2. Let L be a Heyting algebra and $(\nabla, \triangle)$ be a $uq-$pair on L. $(\nabla, \triangle)$ is said to be simple if $\triangle x = 0$ for all $x \in L$, $x \neq 1$.

It is easy to see that $\triangle x = 0$ for all $x \in L$, $x \neq 1$ if and only if $\nabla x = 1$ for all $x \in L$, $x \neq 0$.

Theorem 2.3. *Let $(L, \nabla, \triangle) \in \boldsymbol{MH}$ with more than one element. Then, the following conditions are equivalent:*

(i) *$(L, \nabla, \triangle)$ is simple,*

(ii) *$(\nabla, \triangle)$ is simple.*

Proof. (i) ⇒ (ii): Let $x \in L$, $x \neq 1$. Hence, by Lemma 2.1 we have that $[\triangle x) \in \mathcal{D}(L) \setminus \{1\}$ and taking into account that L is simple we conclude that $[\triangle x) = L$ for all $x \in L$, $x \neq 1$. Therefore, $\triangle x = 0$ for all $x \in L$, $x \neq 1$.
(ii) ⇒ (i): Let $D \in \mathcal{D}(L) \setminus \{1\}$. Then, there is $x \in D$, $x \neq 1$. From the hypothesis and the fact that $\triangle x \in D \setminus \{1\}$, we have that $0 \in D$ which implies that $D = L$. □

3 Deductively Semisimple MH-Algebras

Our next task will be to indicate a characterization of deductively semisimple $MH-$algebras different from the one announced in [5] without proof.

The following result will be used in the subsequent parts of this section.

Lemma 3.1. *In $\boldsymbol{MH}$ the following identities hold true:*

(E12) $\nabla(\triangle x \to \triangle y) = \triangle x \to \triangle y$,

(E13) $\triangle(\triangle x \to \triangle y) = \triangle x \to \triangle y$,

(E14) $\triangle(\triangle x \to y) = \triangle x \to \triangle y$.

Proof.
(E12): We have that

$$\begin{array}{rcll} \nabla(\triangle x \to \triangle y) \wedge \triangle x & = & \nabla((\triangle x \to \triangle y)) \wedge \nabla\triangle x, & \text{by (E11)} \\ & = & \nabla((\triangle x \to \triangle y) \wedge \nabla\triangle x), & \text{by (E3)} \\ & = & \nabla((\triangle x \to \triangle y) \wedge \triangle x), & \text{by (E11)} \\ & = & \nabla(\triangle x \wedge \triangle y), & \text{by (H3)} \\ & = & \nabla\triangle(x \wedge y), & \text{by (E8)} \\ & = & \triangle(x \wedge y), & \text{by (E11)} \\ & = & \triangle x \wedge \triangle y, & \text{by (E8)} \\ & \leq & \triangle y\ . & \end{array}$$

and so, $\nabla(\triangle x \to \triangle y) \leq \triangle x \to \triangle y$. The other inequality follows from (E2).

(E13): It is an immediate consequence of (E12) and (MH2).

(E14): From (E13) and (E7) we have that $\triangle x \to \triangle y = \triangle(\triangle x \to \triangle y) \leq \triangle(\triangle x \to y)$. Conversely, we have that

$$\begin{array}{rcll} \triangle(\triangle x \to y) \wedge \triangle x & = & \triangle(\triangle x \to y) \wedge \triangle\triangle x, & \text{by (E9)} \\ & = & \triangle((\triangle x \to y) \wedge \triangle x), & \text{by (E3)} \\ & = & \triangle(\triangle x \wedge y), & \text{by (H3)} \\ & = & \triangle\triangle x \wedge \triangle y, & \text{by (E8)} \\ & \leq & \triangle y\ . & \end{array}$$

Hence, $\triangle(\triangle x \to y) \leq \triangle x \to \triangle y$. □

In order to find the characterization we were looking for, we define a new binary operation $\Rightarrow$ on an $MH-$algebra L, which we call weak implication, by means of the formula:

$$x \Rightarrow y = \triangle x \to y.$$

Lemma 3.2. *The weak implication verifies the following properties:*

(I1) $x \Rightarrow x = 1$,

(I2) $x \Rightarrow (y \Rightarrow x) = 1$,

(I3) $(x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x \Rightarrow y) \Rightarrow (x \Rightarrow z)) = 1$, (I4) $1 \Rightarrow x = x$,

(I5) $x \Rightarrow \triangle x = 1$,

(I6) $x \leq y$ *implies* $x \Rightarrow y = 1$.

Proof. We will only prove (I3).
(I3): From the definition of $\Rightarrow$, E14 and H12 we have that $(x \Rightarrow y) \Rightarrow (x \Rightarrow z) = \triangle(\triangle x \to y) \to (\triangle x \to z) = (\triangle x \to \triangle y) \to (\triangle x \to z) = (\triangle x \to \triangle y) \to (\triangle x \to z) = \triangle x \to (\triangle y \to z) = x \Rightarrow (y \Rightarrow z)$.

Hence, by (I1) we conclude the proof. ☐

Definition 3.1. Let $L \in \boldsymbol{MH}$ and $D \subseteq L$. D is a week deductive system of L if it satisfies:

(wd1) $1 \in D$,

(wd2) x, $x \Rightarrow y \in D$ imply $y \in D$.

Lemma 3.3. *Let $L \in \boldsymbol{MH}$ and $D \subseteq L$. Then, the following conditions are equivalent:*

(i) *D is monadic deductive system,*

(ii) *D is a weak deductive system.*

Proof. (i) $\Rightarrow$ (ii): Suppose that $x, x \Rightarrow y = \triangle x \to y \in D$. Then, by (D3) we have that $\triangle x \in D$, and so by (D2) we get $y \in D$.
(ii) $\Rightarrow$ (i): It is plain that (D3) follows from (I5) and (wd2). Suppose now that x, $x \to y \in D$. Hence, by (H11) we infer that $x \to y \leq x \Rightarrow y$. From this assertion and (I6) we have that $(x \to y) \Rightarrow (x \Rightarrow y) \in D$, and so by (wd2) we conclude that $y \in D$. ☐

Now, we have achieved our desired goal of characterizing deductively semisimple $MH-$algebras and thus, restated in a more explicit form the results established in [5].

Lemma 3.4. *Let $L \in \boldsymbol{MH}$ and $x, y \in L$. Then, the following identities are equivalent:*

(i) $(\triangle x \to \triangle y) \to \triangle x = \triangle x$,

(ii) $((\triangle x \to \triangle y) \to \triangle x) \to x = 1$,

(iii) $((x \Rightarrow y) \Rightarrow x) \Rightarrow x = 1$.

Proof. (i) $\Rightarrow$ (ii): By (I1), we have $((\triangle x \to \triangle y) \to \triangle x) \to x = \triangle x \to x = x \Rightarrow x = 1$. (ii) $\Rightarrow$ (iii):

$((x \Rightarrow y) \Rightarrow x) \Rightarrow x = \triangle(\triangle(\triangle x \to y) \to x) \to x$. Hence, by (E14) we have $((x \Rightarrow y) \Rightarrow x) \Rightarrow x = (\triangle(\triangle x \to y) \to \triangle x) \to x = ((\triangle x \to \triangle y) \to \triangle x) \to x = 1$.

(iii) $\Rightarrow$ (i): From the hypothesis, (I2) and (I6) we conclude that $(x \Rightarrow y) \Rightarrow x = x$ and so, $x = \triangle(\triangle x \to y) \to x$. Hence, by (E14) we have that $\triangle x = \triangle(\triangle x \to y) \to \triangle x = (\triangle x \to \triangle y) \to \triangle x$. ☐

Theorem 3.5. *Let $L \in \boldsymbol{MH}$ be non trivial. Then, the following conditions are equivalent:*

(i) *L is a deductively semisimple $MH-$algebra,*

(ii) *L satisfies the identity* $(\triangle x \to \triangle y) \to \triangle x = \triangle x$,

(iii) *$\triangle(L)$ is a monadic Tarski algebra,*

(iv) *$\triangle(L)$ is a monadic Boolean algebra.*

Proof. It is a direct consequence of Lemma 3.1, 3.2, 3.3, 3.4, [11, pag. 427-431] (see also [12]) and well-known results on monadic Tarski algebras (see [13]) and monadic Boolean algebras (see [9]). ☐

4 Concluding Remarks

Recall that a generalized Heyting algebra (or H_0-algebra)([2]) is an algebra $\langle L, \wedge, \vee, \rightarrow, 1\rangle$ of type $(2, 2, 2, 0)$ which satisfies (H1) to (H5). Then, we can define monadic generalized Heyting algebras (or MH_0-algebras) as triples $(L, \nabla, \triangle)$ such that L is an H_0-algebra and it verifies the identities (E2), ..., (E11).

Hence, all the results obtained for $MH-$algebras which do not involve the first element also hold true for MH_0-algebras. Thus, for example, semisimple MH_0-algebras are characterized as follows:

Let L be a non trivial MH_0-algebra, then the following conditions are equivalent:

(i) L is a deductively semisimple MH_0-algebra,

(ii) L satisfies the identity $(\triangle x \rightarrow \triangle y) \rightarrow \triangle x = \triangle x$,

(iii) $\triangle(L)$ is a Tarski algebra.

On the other hand, if $L \in \boldsymbol{MH}$ it is easy to see that the identity (E1) is a consequence of (E7) and (E11). However, condition (E13) is not a consequence of (E6),..., (E9). To this end, let us consider the lattice L shows in Fig. 1 with the operations $\rightarrow$ and $\triangle$ defined as follows:

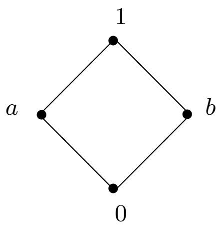

Fig. 1. Lattice L

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

x	$\triangle x$
0	0
a	a
b	0
1	1

Indeed, it is easy to see that (E6), ..., (E9) hold, but (E13) is not verified for $x = a$ and $y = 0$. Motivated in this example, we introduce the notion of $U-$Heyting algebras as algebras $\langle L, \wedge, \vee, \rightarrow, \triangle, 0, 1\rangle$ of type $(2, 2, 2, 1, 0, 0)$ such that $\langle L, \wedge, \vee, \rightarrow, 0, 1\rangle$ is a Heyting algebra and $\triangle$ satisfies (E6),...,(E9) and (E13). In a prospective paper we will develop the theory of these algebras.

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Competing Interests

The authors declare that no competing interests exist.

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