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The Characterization of the Cubic Rank Inverse Weibull Distribution

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Authors' contributions

This work was carried out in collaboration among all authors. All authors designed the study, performed the statistical analysis, wrote the protocol, wrote the first draft of the manuscript, managed the analyses of the study and managed the literature searches. All authors read and approved the final manuscript.

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Abstract

This work provides a new statistical distribution named Cubic rank transmuted Inverse Weibull distribution which was developed using the cubic transmutation map. Various statistical properties of the new distribution which includes: hazard function, moments, moment generating function, skewness, kurtosis, Renyl entropy and the order statistics were studied. A maximum likelihood estimation method was used in estimating the parameters of the distribution. Applications to real data set show the tractability of the distribution over other distributions and its sub-model.

Keywords: Cubic transmutation map; Renyl entropy; moment; kurtosis; order statistics; kurtosis.

1 Introduction

Standard probability distribution has limited application most especially in the area that concern the modeling of life time data which sometime is unimodal or non-monotone (bathtub shape), bimodal (bathtub with accident hump), thus there is need to develop a flexible distribution that is tractable and can effectively be used in modeling life time distribution. Several methods have been developed to address this problem but

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not limited to the work of Eugene N, [1], Aslam M [2], Celic and Gulokksuz [3], Cordeiro et al. [4], Gupta [5-7], Nofal et al. [8], Granzotto et al. [9], Shaw WT et al. [10], Merovic F et al. [11] etc.

The inverse Weibull distribution which also known as the frechet distribution has been used to model the failure characteristics of mechanical components such as, crankshaft, jet engines, pistons etc. inverse Weibull (IW) distribution found usefulness in many areas including, reliability, biological processes and branching as contained in the work of keller and kamath [12]. The author used the Inverse Weibull (IW) to model the degradation of phenomena of mechanical components such as piston diesel engine and crankshaft. Several authors have investigated the properties of IW distribution, and this does not limited to the work of Oluyede and Yang [13], Kersey and Oluyede [14], Parai et al. [15], Bera Walter T. [16], Ramos PL, et al. [10,17], Louzada F, et al. [8] etc.

The cumulative distribution function (cdf) of IW distribution is given by

$$\bar{H}(x;\alpha,\eta) = \exp\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right) \qquad x \geq 0, \alpha, \eta > 0 \tag{1}$$

The associated probability density function (pdf) is given by

$$\bar{h}(x;\alpha,\eta) = \eta\alpha^{\eta} \exp\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right) \qquad x \geq 0, \alpha, \eta > 0 \tag{2}$$

And the hazard function is given by

$$\psi(x) = \frac{\eta\alpha^{\eta} \exp\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}{\exp\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} \tag{3}$$

Where α, η are the scale and shape parameters respectively.

This work is meant to introduce a new distribution be called Cubic rank transmuted inverse Weibull (CRTIW) distribution which can be can be employed in modelling real data which exhibits increasing, decreasing, bathtub (non-monotone) failure rates by using cubic rank transmutation map (CRTM) proposed by Granzotto et al. [9].

2 Methodology

2.1 Cubic rank transmuted Kumaraswamy (CRTIW) distribution

Let T be a random variable having a CRTIW distribution with $\alpha, \eta, \lambda_1, \lambda_2$ parameters. The cdf and the pdf are respectively given by

$$F(x) = \lambda_1\bar{H}(x) + (\lambda_2 - \lambda_1)\bar{H}^2(x) + (1 - \lambda_2)\bar{H}^3(x)$$

And

$$f(x) = \bar{h}(x)[\lambda_1 + 2(\lambda_2 - \lambda_1)\bar{H}(x) + 3(1 - \lambda_2)\bar{H}^2(x)]$$

Where $\bar{H}(x)$ and $f(x)$ are the cdf and the pdf of the baseline distribution. $G(x)$ is the cdf of the cubic rank transmuted G ((CRT-G), $\bar{h}(x)$ is the pdf of the cubic rank transmuted G (CRT-G) and $\lambda_1 \in [0,1]$, $\lambda_2 \in [-1,1]$. Granzotto et al. [9] have suggested cubic rank transmuted Weibull, cubic rank transmuted log-logistic distributions etc. Bugra Saracoglu et al. [18], studied the properties of cubic rank transmuted kumaraswamy distribution, Cubic transmuted Gompertz distribution was studied by Ogunde AA et al. [19] etc.

The cumulative density function (cdf) and the probability density function (pdf) of this distribution are respectively given by

$$F(x) = e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + (\lambda_2 - \lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + (1-\lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\} \tag{4}$$

And

$$f(x) = \eta\alpha^{\eta}x^{-(\eta+1)}e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + 2(\lambda_2 - \lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\} \tag{5}$$

The graph of the cdf and the pdf are respectively drawn below taken α= $a, b = \eta, \lambda_1 = b_1 and\ \lambda_2 = b_2$ for various values of the parameters of the distribution.

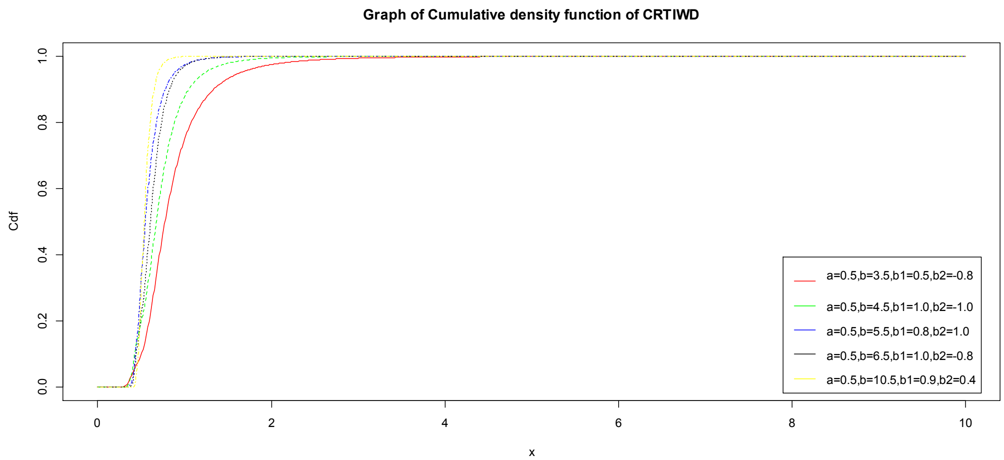

Fig. 1. The graph of the cdf of CRTIW distribution

✓ The Fig. 1 drawn above shows that the cdf of CRTIW distribution has a proper pdf.

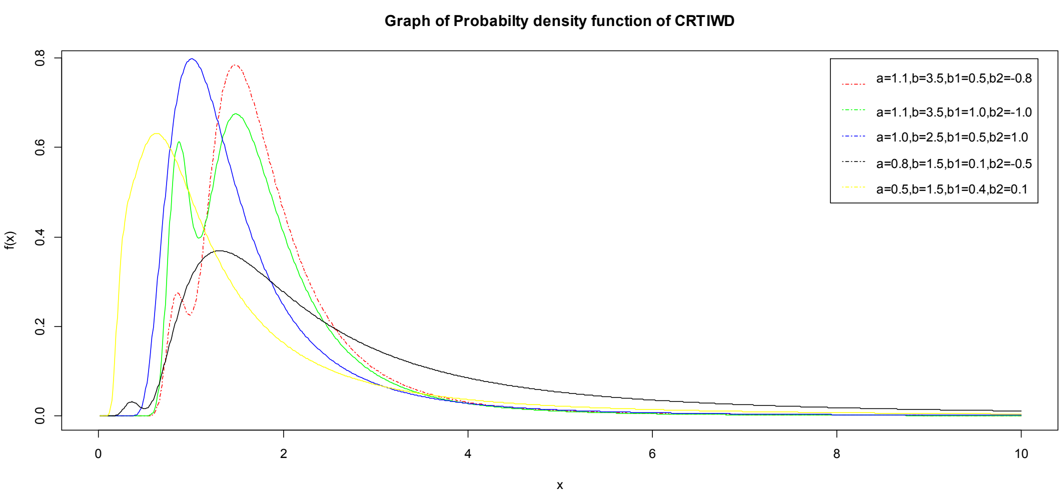

Fig. 2. The graph of the pdf of CRTIW distribution

✓ The pdf graph of CRTIW drawn above indicates that the distribution can be used to model real life data that are bi-modal in nature.

Reliability function and the hazard function (hf) of the CRTIW $(\alpha, \eta, \lambda_1, \lambda_2)$ distribution are given in equation (6) and (7), respectively

$$R(x) = 1 - F(x)$$

$$R(x) = 1 - e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + (\lambda_2 - \lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + (1 - \lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\} \tag{6}$$

$$h(x) = \frac{f(x)}{R(x)}$$

$$h(x) = \frac{\eta\alpha^{\eta}e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + 2(\lambda_2 - \lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + 3(1 - \lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\}}{1 -^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + (\lambda_2 - \lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + (1 - \lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\}} \tag{7}$$

The graph of the hf of CRTIW distribution is drawn below for various values of the parameters of the distribution.

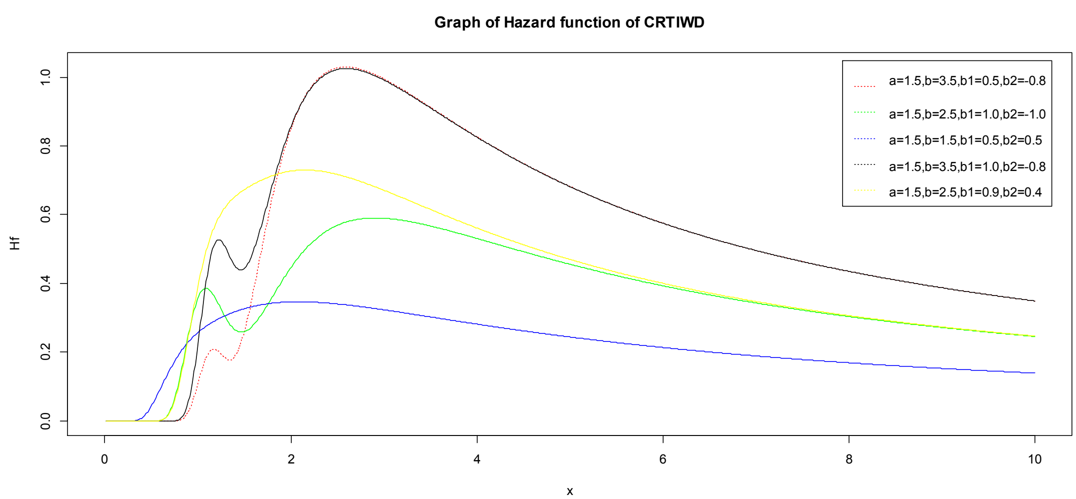

Fig. 3. The graph of the hazard function of CRTIW distribution

- ✓ The Fig. 3 drawn above indicates that the CRTIW distribution has inverted bathtub shape with sum having incidence of accident hump.

3 Moments of CRTIW Distribution

Let X be a random variable having a CRTIW distribution with $\alpha, \eta, \lambda_1, \lambda_2$ parameters k^{th}.

Moment of this random variable is obtained as follows:

$$E(X^k) = \int_{-\infty}^{\infty} X^k f(x)dx \tag{8}$$

$$E(X^k) = \eta\alpha^\eta \int_{-\infty}^{\infty} x^{k-(\eta+1)} e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)} \left\{\lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)}\right]^2\right\} dx \tag{9}$$

Equation (9) can be splitted into

$$\Lambda_1 = \eta\alpha^\eta\lambda_1 \int_{-\infty}^{\infty} x^{k-(\eta+1)} e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)} dx = \lambda_1 \alpha^r \Gamma\left(1 - \frac{k}{\eta}\right) \tag{10}$$

$$\Lambda_2 = 2\eta\alpha^\eta(\lambda_2 - \lambda_1) \int_{-\infty}^{\infty} x^{k-(\eta+1)} \left[e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)}\right]^2 dx = 2^{\frac{k}{\eta}} \alpha^k (\lambda_2 - \lambda_1)\Gamma\left(1 - \frac{k}{\eta}\right) \tag{11}$$

$$\Lambda_3 = 3\eta\alpha^\eta(1 - \lambda_2) \int_{-\infty}^{\infty} x^{k-(\eta+1)} \left[e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)}\right]^{32} dx = 3^{\frac{k}{\eta}} \alpha^k (1 - \lambda_2)\Gamma\left(1 - \frac{k}{\eta}\right) \tag{12}$$

Then

$$E(X^k) = \Lambda_1 + \Lambda_2 + \Lambda_3 \tag{13}$$

Finally we have

$$E(X^k) = \alpha^k \Gamma\left(1 - \frac{k}{\eta}\right)\left[\lambda_1 + 2^{\frac{k}{\eta}}(\lambda_2 - \lambda_1) + 3^{\frac{k}{\eta}}(1 - \lambda_2)\right] \tag{14}$$

And expression for the $1^{st}, 2^{nd}, 3^{rd} and\ 4^{th}$ moments can be obtained by letting $k = 1,2,3\ and\ 4$.

$$E(X) = \alpha\Gamma\left(1 - \frac{1}{\eta}\right)\left[\lambda_1 + 2^{\frac{1}{\eta}}(\lambda_2 - \lambda_1) + 3^{\frac{1}{\eta}}(1 - \lambda_2)\right] \tag{15}$$

$$E(X^2) = \alpha^2\Gamma\left(1 - \frac{2}{\eta}\right)\left[\lambda_1 + 2^{\frac{2}{\eta}}(\lambda_2 - \lambda_1) + 3^{\frac{2}{\eta}}(1 - \lambda_2)\right] \tag{16}$$

$$E(X^3) = \alpha^3\Gamma\left(1 - \frac{3}{\eta}\right)\left[\lambda_1 + 2^{\frac{3}{\eta}}(\lambda_2 - \lambda_1) + 3^{\frac{3}{\eta}}(1 - \lambda_2)\right] \tag{17}$$

$$E(X^4) = \alpha^4\Gamma\left(1 - \frac{4}{\eta}\right)\left[\lambda_1 + 2^{\frac{4}{\eta}}(\lambda_2 - \lambda_1) + 3^{\frac{3}{\eta}}(1 - \lambda_2)\right] \tag{18}$$

And the variance is

$$Var(X) = E(x^2) - [E(x)]^2$$

Then an expression for the variance of X is given by

$$Var(X) = \alpha^2\Gamma\left(1 - \frac{2}{\eta}\right)\left[\lambda_1 + 2^{\frac{2}{\eta}}(\lambda_2 - \lambda_1) + 3^{\frac{2}{\eta}}(1 - \lambda_2)\right]$$
$$- \left[\alpha\Gamma\left(1 - \frac{1}{\eta}\right)\left[\lambda_1 + 2^{\frac{1}{\eta}}(\lambda_2 - \lambda_1) + 3^{\frac{1}{\eta}}(1 - \lambda_2)\right]\right]^2 \tag{19}$$

The n^{th} central moments, J_n of the CRTIW distribution can be obtained from the k^{th} non-central moments from the relation

$$J_n = E(X - E(X))^2 = \sum_{k=0}^{n} \binom{n}{k} (-\mu)^{n-k} E(X^k)$$

Thus the n^{th} central moment of the CRTIW distribution is given by

$$J_n = \sum_{k=0}^{n} \binom{n}{k} (-\mu)^{n-k} \alpha^k \Gamma\left(1 - \frac{k}{\eta}\right) \left[\lambda_1 + 2^{\frac{k}{\eta}}(\lambda_2 - \lambda_1) + 3^{\frac{k}{\eta}}(1 - \lambda_2)\right] \tag{20}$$

3.1 Coefficient of variation, Skewness and Kurtosis

The coefficient of variation (CV), skewness (CS) and kurtosis (CK) for CRTIW distribution can be obtained by using equation (21), (22) and (23) respectively

$$CV = \left(\frac{\phi_2}{\phi_1^2} - 1\right)^{\frac{1}{2}} \tag{21}$$

$$CS = \frac{\phi_3 - 3\phi_2\phi_1^2 + 2\phi_1^3}{(\phi_2 - \phi_1^2)^{\frac{3}{2}}} \tag{22}$$

$$CK = \frac{\phi_4 - 4\phi_2\phi_3 + 2\phi_1^4}{(\phi_2 - \phi_1^2)^{\frac{3}{2}}} \tag{23}$$

Where

$$\phi_i = \alpha^i \Gamma\left(1 - \frac{i}{\eta}\right) \left[\lambda_1 + 2^{\frac{i}{\eta}}(\lambda_2 - \lambda_1) + 3^{\frac{i}{\eta}}(1 - \lambda_2)\right], i = 1,2,3,4$$

Table 1 drawn below summaries the values for the first moment, the second moment, variance, coefficient of variation, kurtosis, and skewness for various parameters values of the CRTIW distribution with $(\alpha, \eta, \lambda_1, \lambda_2)$ parameters.

Table 1. Mean, variance, coefficient of variation, skewness and kurtosis for various parameter values of CRTIW distribution

Parameter values $(\boldsymbol{\alpha}, \boldsymbol{\eta}, \boldsymbol{\lambda_1}, \boldsymbol{\lambda_2})$	$\boldsymbol{E(X)}$	$\boldsymbol{E(X^2)}$	$\boldsymbol{Var(X)}$	$\boldsymbol{CV}$	$\boldsymbol{CS}$	$\boldsymbol{CK}$
0.1,5,0.3,0.4	0.1353	0.0203	0.0020	1.054	3.5090	44.0057
2,10,0.5,0.6	2.2518	5.1839	0.1133	14.948	1.6454	9.7202
5,10,0.8,1.0	5.4198	29.7713	0.3971	11.627	16.3819	-199.611
5,10,1.0,-1.0	5.8171	34.7542	0.9155	16.448	1.2714	14674.62
0.5,10,1.0,-1.0	0.3475	0.2139	0.0091	27.452	0.1648	7.4439

3.2 Moment generating function of CRTIW distribution

Let X be a random variable from CRTIW distribution$(\alpha, \eta, \lambda_1, \lambda_2)$. The moment generating function of CRTIW distribution, $M_x(x)$, is obtained as follows;

$$M_x(x) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k) \tag{24}$$

Then we have

$$M_x(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \alpha^k \Gamma\left(1 - \frac{k}{\eta}\right)\left[\lambda_1 + 2^{\frac{k}{\eta}}(\lambda_2 - \lambda_1) + 3^{\frac{k}{\eta}}(1 - \lambda_2)\right] \tag{25}$$

3.3 Renyl entropy

The Renyl entropy of a random variable S represents a measure of uncertainty. A large value of entropy indicates the greater level of uncertainty in the data. The Renyl, A. [20], introduced the Renyl entropy defined as

$$h_{Ⲗ} = \frac{1}{Ⲗ - 1} log \int_{-\infty}^{\infty} f^{Ⲗ}(x)dx\,, \quad Ⲗ > 1, Ⲗ = 0 \tag{26}$$

$$h_{Ⲗ} = \frac{1}{Ⲗ - 1} log \int_{-\infty}^{\infty} \left[\eta\alpha^\eta x^{-(\eta+1)} e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)} \left\{\lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)} + 3(1 - \lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)}\right]^2\right\}\right]^{Ⲗ} dx$$

By letting

$$\Gamma_1 = \int_{-\infty}^{\infty} \left\{\eta\alpha^\eta x^{-(\eta+1)} e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)}\right\}^{Ⲗ} = \eta^{Ⲗ-1} Ⲗ^{\frac{Ⲗ(\eta+1)+1}{\eta}} \alpha^{Ⲗ(\eta+1)} \Gamma\left\{1 - \frac{Ⲗ(\eta + 1)}{\eta}\right\} \tag{27}$$

$$\Gamma_2 = \int_{-\infty}^{\infty} \left\{\lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)} + 3(1 - \lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^\eta\right)}\right]^2\right\}^{Ⲗ} dx$$

Then the Renyl entropy CRTIW is given by

$$h_{Ⲗ} = \frac{1}{Ⲗ - 1} log\left\{\eta^{Ⲗ-1} Ⲗ^{\frac{Ⲗ(\eta+1)+1}{\eta}} \alpha^{Ⲗ(\eta+1)} \Gamma\left\{1 - \frac{Ⲗ(\eta + 1)}{\eta}\right\} * \Gamma_2\right\} \tag{28}$$

The Table 2 drawn below gives the values of entropy of CRTIW distribution for different values of the parameters and keeping the values of $\alpha = 1.5\ and\ \eta = 3.5$.

Table 2. Values of Renyl entropy of CRTIW distribution for different parameters values

$\boldsymbol{\lambda_1, \lambda_2}$	$Ⲗ = \mathbf{2}$	$Ⲗ = \mathbf{4}$	$Ⲗ = \mathbf{10}$	$Ⲗ = \mathbf{20}$	$Ⲗ = \mathbf{100}$
0.1,0.1	−0.9304	−0.7453	−0.6029	−0.5250	−0.4656
0.5,0.5	−0.8921	−0.7253	−0.5956	−0.5457	−0.5055
1.0, −1.0	−1.1239	−0.9855	−0.8693	0.8366	−0.7932
−1.0, −1.0	−0.2120	−0.3321	−0.2863	−0.2434	−0.2756
1.0,1.0	−0.5913	−0.4087	−0.2677	−0.2063	−0.1750

- ✓ From the table drawn above it can be discovered that the values of entropy approaches zero as the value of $Ⲗ$ increases.

4 Distribution of Order Statistics for CRTIW Distribution

Let $X_1, X_2, . . . , X_n$ be a random sample taken from CRTIW with parameters $\alpha, \eta, \lambda_1, and\ \lambda_2$ and $x_1, x_2, . . . , x_n$ denote the order statistics of the random sample. The pdf of the i^{th} order statistics $f_{X_{(i)}}$, can be obtained as follows.

$$f_{X_{(i)}}(x) = \mathrm{V} f_X(x)[F_X(x)]^{i-1}[1-F_X(x)]^{n-i} \tag{29}$$

$$f_{X_{(i)}}(x) = \mathrm{V}\eta\alpha^{\eta}x^{-(\eta+1)}e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\}$$

$$\times\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + (\lambda_2-\lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + (1-\lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\}\right]^{i-1}$$

$$\times\left[1-e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + (\lambda_2-\lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + (1-\lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\}\right]^{n-i} \tag{30}$$

Where, $i = 1,2,...,n$ and $\mathrm{V} = n!\,[(i-1)!\,(n-1)!]^{-1}$.

The pdf of the first order statistics is given by

$$f_{X_{(1)}}(x) = \eta\alpha^{\eta}x^{-(\eta+1)}e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\}$$
$$\times\left[1-e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + (\lambda_2-\lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + (1-\lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\}\right]^{n-1} \tag{31}$$

The pdf of the n^{th} order statistics is given by

$$f_{X_{(n)}}(x) = \eta\alpha^{\eta}x^{-(\eta+1)}e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\}$$
$$\times\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + (\lambda_2-\lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + (1-\lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\}\right]^{n-1} \tag{32}$$

5 Maximum Likelihood Estimation for Parameters of CRTIW Distribution

Let $X_1, X_2, ..., X_n$ be a random sample taken from CRTIW with parameters $\mathcal{Z}(\alpha, \eta, \lambda_1, \lambda_2)$. The likelihood function is given by

$$L(\mathcal{Z}\,|\,\underline{x}) = \prod_{i=1}^{n}\eta\alpha^{\eta}x^{-(\eta+1)}e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\left\{\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\alpha}{x}\right)^{\eta}\right)}\right]^2\right\} \tag{33}$$

And the log-likelihood function is given by

$$l(\mathcal{Z}\,|\,\underline{x}) =$$
$$nlog(\eta) + n\eta\log(\alpha) - (\eta+1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}log\{\lambda_1 + 2(\lambda_2-\lambda_1)\kappa_i + 3(1-\lambda_2){\kappa_i}^2\} \tag{34}$$

$$\kappa_i = e^{\left(-\left(\frac{\alpha}{x_i}\right)^{\eta}\right)}$$

Then the score vector $\nabla l = \frac{\delta l}{\delta \alpha}, \frac{\delta l}{\delta \eta}, \frac{\delta l}{\lambda_1}, \frac{\delta l}{\lambda_2}$ has components,

$$\frac{\delta l}{\delta \alpha} = \frac{n\eta}{\alpha} - \frac{2\sum_{i=1}^{n} \kappa_i \alpha^{\eta-1} \eta x^{-\eta}\{(\lambda_2 - \lambda_1) + 3(1-\lambda_2)\kappa_i\}}{\sum_{i=1}^{n} \kappa_i\{\lambda_1 + 2(\lambda_2 - \lambda_1) + 3(1-\lambda_2)\kappa_i\}} \quad (35)$$

$$\frac{\delta l}{\delta \eta} = \frac{n}{\eta} + n\log(\alpha) + \sum_{i=1}^{n} \log(x_i) + \frac{2\sum_{i=1}^{n} \kappa_i \left(\frac{\alpha}{x}\right)^{\eta} log\left(\frac{\alpha}{x}\right)\{(\lambda_2 - \lambda_1) + 3(1-\lambda_2)\kappa_i\}}{\sum_{i=1}^{n} \kappa_i\{\lambda_1 + 2(\lambda_2 - \lambda_1) + 3(1-\lambda_2)\kappa_i\}} \quad (36)$$

$$\frac{\delta l}{\lambda_1} = -\frac{\sum_{i=1}^{n} \kappa_i}{\sum_{i=1}^{n} \kappa_i\{\lambda_1 + 2(\lambda_2 - \lambda_1) + 3(1-\lambda_2)\kappa_i\}} \quad (37)$$

$$\frac{\delta l}{\lambda_2} = \frac{\sum_{i=1}^{n} \kappa_i\,(2 - 3\kappa_i)}{\sum_{i=1}^{n} \kappa_i\{\lambda_1 + 2(\lambda_2 - \lambda_1) + 3(1-\lambda_2)\kappa_i\}} \quad (38)$$

These non-linear equations can be solved by using the Newton-Raphson method. Then the MLEs for α, η, λ_1 and λ_2 parameters are obtained.

6 Application to Real Life Data

In this section, we made use of two real data sets to show the applicability of CRTIW distribution in modeling real life data. The first real data set has been obtained by Dasgupta [21]. This dataset is related to holes operation on jobs made of iron sheet. This dataset is as follows: 0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26,0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16,0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16,0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16. This data set has been fitted to CRTIW, Cubic rank transmuted kumaraswamy (CRTKw) distribution, Kumaraswamy Weibull (Kw) distribution, transmuted Rayleigh (TR) distribution, transmuted Weibull (TW) distribution, exponentiated exponential (EE) distribution and Inverse Weibull (IW) distribution. In order to know how this distribution fit to the data set, measures such as Anderson Darling goodness-of-fit statistics A^*, Kolmogorov-Smirnoff (KS) statistics, Akaike Information criterion (AIC), Bayesian information criterion (BIC) and -2*log-likelihood will be obtained. These measures are given by

$$A^* = -n - \frac{1}{n}\sum_{v=1}^{n}(2v-1)log\left[F\left(X_{(v)}\right)\left(1 - F\left(X_{(n-v+1)}\right)\right)\right] \quad (39)$$

$$KS = sup(|F(x) - F_n(n)|) \quad (40)$$

$$AIC = -2(l + j) \quad (41)$$

$$AICc = AIC + \left(\frac{2j(j+1)}{n-j-1}\right) \quad (42)$$

$$BIC = -2l + jlog(n) \quad (43)$$

Where $X_{(v)}$ is the v^{st} order statistics, v is trhe number of parameters and n is the sample size. l is the value of log-likelihood function and $F_n(n)$ is the empirical distribution function.

The density function of the fitted distribution is given by

$$CRTKw: f(x) = ab(1-x^a)^{b-1}[\lambda_1 + 2(\lambda_2 - \lambda_1)(1-x^a)^b + 3(1-\lambda_2)((1-x^a)^b)^2]$$
$$a, b > 0, x \in [0,1], \lambda_1 \in [0,1], \lambda_2 \in [-1,1]$$

$$TKw: f(x) = ab(1-x^a)^{b-1}[1-\lambda+2\lambda(1-x^a)^b], \qquad a,b>0, x\in[0,1], \lambda\in[-1,1]$$

$$TR: f(x) = \frac{x}{\alpha^2}e^{-\frac{x^2}{2\alpha^2}}\left[1-\lambda+2\lambda e^{-\frac{x^2}{2\alpha^2}}\right],\ \ \alpha,x>0$$

$$EE: f(x) = \alpha\lambda\left(1-e^{-\lambda x}\right)^{\alpha-1}e^{-\lambda x},\ \ \alpha,\lambda,x>0$$

Maximum likelihood estimates for the unknown parameters of these distribution and their standard errors are given in Tables 3 and 4. Plots of the Total Test of Time (TTT) plot and the empirical density are provided in Figs. 4 and 5 respectively.

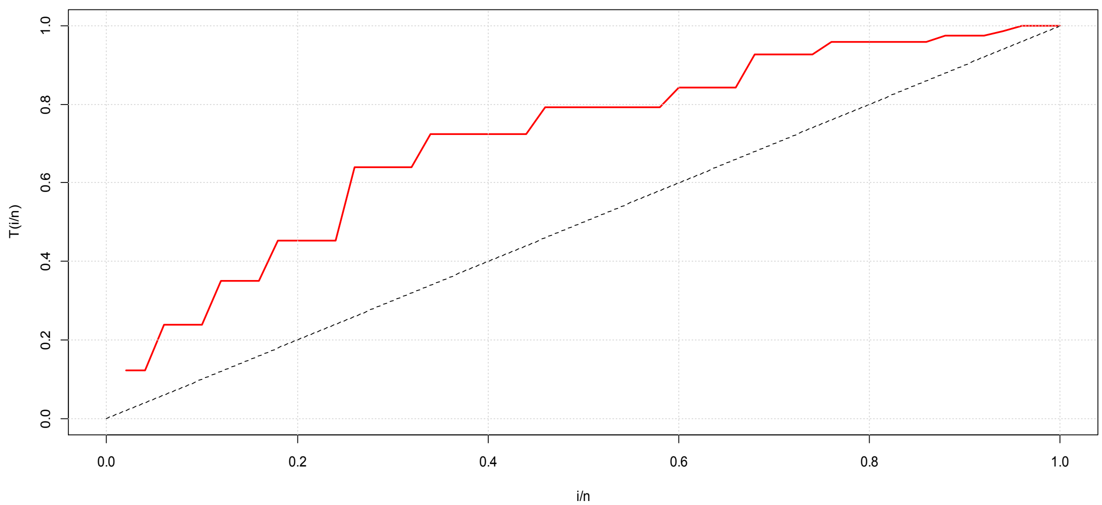

Fig. 4. Graph of the TT plot of the operation data

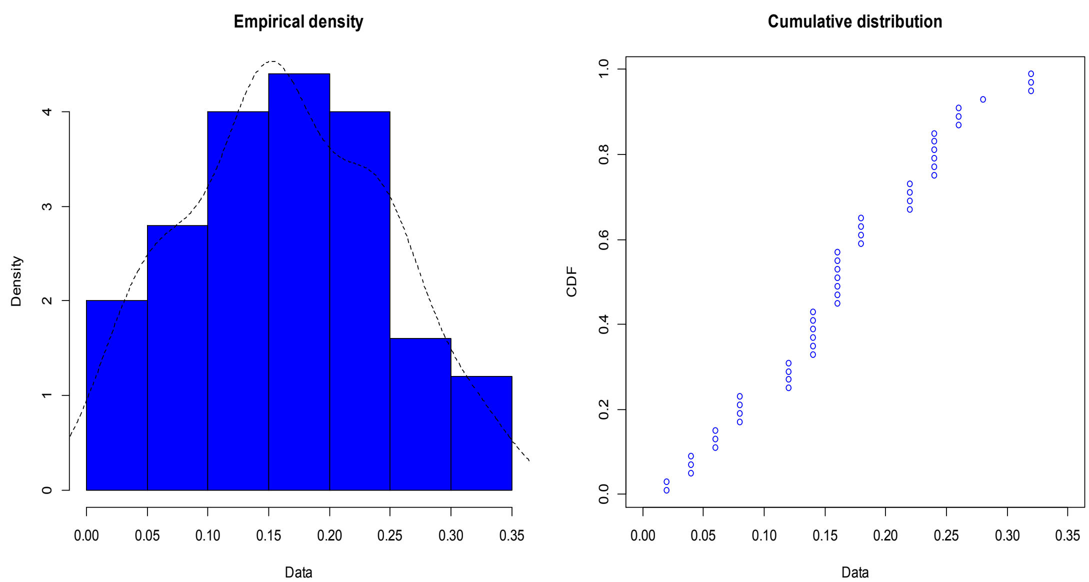

Fig. 5. Graph of the empirical density function and cumulative density function of operation data

Table 3. Parameter estimate and standard error in parenthesis for operation data set

Distribution	ML estimates			
CRTIW	0.278	0.754	38.143	10.018
$(\hat{\alpha}, \hat{\eta}, \hat{\lambda}_1, \hat{\lambda}_2)$	(NaN)	(0.061)	(NaN)	(NaN)
CRTKw	1.898	38.242	0.994	−0.576
$(\hat{a}, \hat{b}, \hat{\lambda}_1, \hat{\lambda}_2)$	(0.2604)	(15.580)	(0.399)	(0.844)
TKw	1.934	30.186	−0.291	−
$(\hat{a}, \hat{b}, \hat{\lambda})$	(0.352)	(13.792)	(0.441)	(−)
EE	3.172	11.360	−	−
$(\hat{\alpha}, \hat{\lambda})$	(0.707)	(1.568)	(−)	(−)
TR	0.121	−0.265	−	−
$(\hat{\alpha}, \hat{\lambda})$	(0.011)	(0.309)	(−)	(−)
IW	0.095	1.236	−	−
$(\hat{\alpha}, \hat{\eta})$	(0.012)	(0.118)	(−)	(−)

Table 4. Measurements of goodness of fit for operation data set

Distribution	$-2log$	AIC	BIC	$AICc$	A^*	KS	PV
$CRTIW$	−339.988	−331.988	−324.340	−331.099	NaN	9.310	2.2e-16
$CRTKw$	−112.502	−106.240	−99.580	−97.457	0.473	0.114	0.5359
TKw	−112.502	−10.502	−100.766	−105.980	0.625	0.105	0.6375
EE	−104.572	−100.572	−96.748	−100.317	1.261	0.110	0.5815
TR	−112.117	−108.117	−104.293	−107.862	2.004	0.107	0.616
IW	−72.448	−70.449	−66.625	−70.193	3.567	0.233	0.009

The second dataset is about the total milk production in the first birth of 107 cows living in Carnauba farm in Brazil. This data exist in the work of Cordeiro and Brito [1] and Brito [22] and it has also been used by Yousof et al. [23], Bugra S et al. [18]. The data set is given has follows: 0.4365,0.4260,0.5140,0.6907,0.7471, 0.2605, 0.6196, 0.8781, 0.4990, 0.6058, 0.6891, 0.5770, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.4800, 0.5707, 0.7131, 0.5853, 0.6768, 0.5350, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.3480, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.7290, 0.0168, 0.5529, 0.4530, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.6750, 0.5113, 0.5447, 0.4143, 0.5627, 0.5150, 0.0776, 0.3945, 0.4553, 0.4470, 0.5285, 0.5232, 0.6465, 0.0650, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188,0.2160, 0.6707, 0.6220, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694,0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.6860, 0.0609, 0.6488, 0.2747. This dataset has been fitted to CRTIW, Cubic rank transmuted kumaraswamy (CRTKw) distribution, Transmuted Kumaraswamy (TKw) distribution, Exponentiated Kumaraswamy (EKw) distribution and Inverse Weibull (IW) distribution and their pdf is given by;

$$CRTKw: f(x) = ab(1-x^a)^{b-1}[\lambda_1 + 2(\lambda_2 - \lambda_1)(1-x^a)^b + 3(1-\lambda_2)((1-x^a)^b)^2]$$

$$a, b > 0, x \in [0,1], \lambda_1 \in [0,1], \lambda_2 \in [-1,1]$$

$$TKw: f(x) = ab(1-x^a)^{b-1}[1-\lambda + 2\lambda(1-x^a)^b], \qquad a, b > 0, x \in [0,1], \lambda \in [-1,1]$$

$$EKw: f(x) = \alpha abx^a(1-x^a)^{b-1}[(1-x^a)^{b-1}]^{\alpha-1}, \qquad a, b, \alpha > 0, x \in [0,1]$$

The maximum likelihood estimates of this distributions and their standard error in parenthesis are given in Table 5 and Table 6 Plots of the Total Test of Time (TTT) plot and the empirical density are provided in Figs. 6 and 7 respectively. Table 5 Parameter estimate and standard error in parenthesis for milk production.

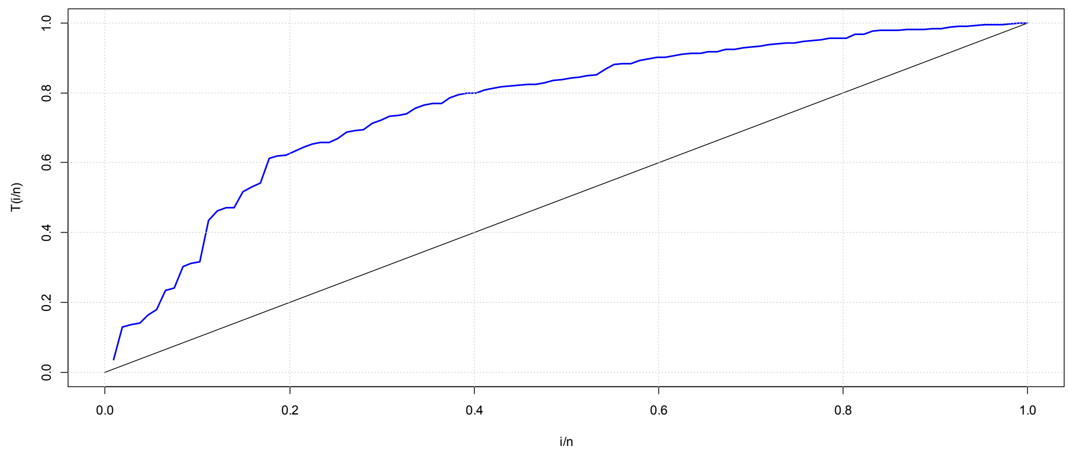

Fig. 6. Graph of the TTT plot of milk production data

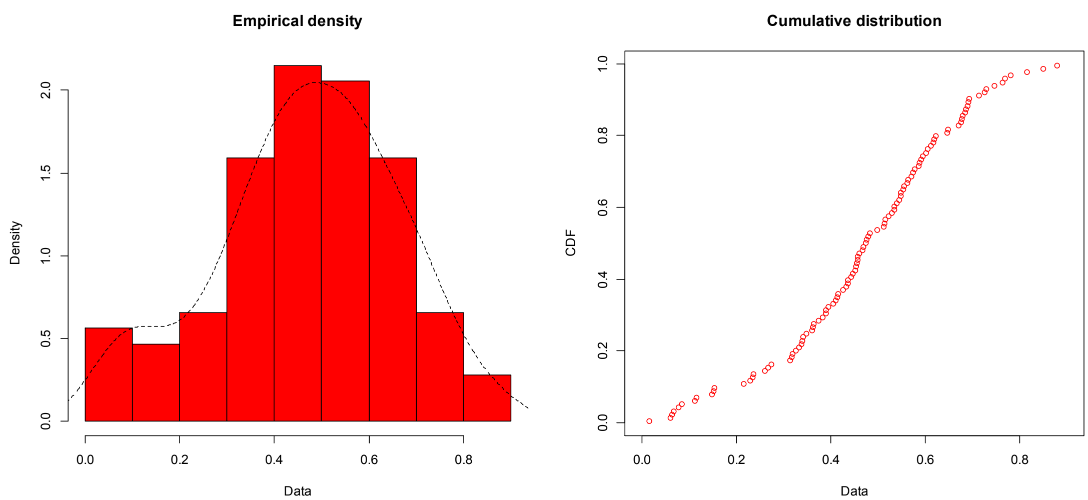

Fig. 7. Graph of the empirical density and the cumulative distribution of milk production data

Table 5. Parameter estimate and standard error in parenthesis for milk data

Distribution	ML estimates			
CRTIW	0.055	1.715	0.906	−1.578
$(\hat{\alpha},\hat{\eta},\hat{\lambda}_1,\hat{\lambda}_2)$	(0.005)	(0.140)	(0.311)	(0.527)
CRTKw	1.535	3.923	0.634	−0.801
$(\hat{a},\hat{b},\hat{\lambda}_1,\hat{\lambda}_2)$	(0.188)	(0.565)	(0.209)	(0.475)
TKw	1.823	3.436	−`0.561	−
$(\hat{a},\hat{b},\hat{\lambda})$	(0.274)	(0.562)	(0.225)	(−)
EKw	0.5315	7.140	2.195	−
$(\hat{a},\hat{b},\hat{\alpha})$	(1.870)	(3.092)	(0.222)	(−)
IW	0.095	1.236	−	−
$(\hat{\alpha},\hat{\eta})$	(0.012)	(0.118)	(−)	(−)

Table 6. Measurement of goodness of fit for milk production

$Distribution$	$-2log$	AIC	BIC	$AICc$	A^*	KS	PV
$CRTIW$	−89.798	−81.798	−74.150	−80.909	1.950	0.168	0.117
$CRTKw$	−62.223	−54.223	−43.532	−53.831	0.1176	0.0427	0.9899
TKw	−54.097	−48.097	−40.079	−47.864	0.636	0.060	0.836
EKw	−55.114	−49.114	−41.443	−48.881	0.595	0.075	0.587
IW	−74.448	−70.449	−66.624	−70.193	3.567	0.233	0.009

7 Conclusion

In this work, we have introduced a new distribution (model) called cubic rank transmuted inverse Weibull distribution by using the cubic rank transmutation map developed by Granzotto et al. [9]. Some statistical properties of this new distribution were investigated and the maximum likelihood estimators for unknown parameters of thus distribution were obtained.

In the real data analysis, the dataset from Dasgupta R [21] was considered first and from Table 3 shows that the CRTIW distribution has the best fit because it possesses the smallest AIC, BIC, AICc and -2log-likelihood among all fitted models and the same is also applicable for the data set of Cordeiro and Don Santos Brito [1], Brito RS. [22] and Bugra et al. [18], has presented in Table 6. Based on our findings, CRTIW distribution can be used to model the two data sets. The CRTIW distribution can be used to model data that are non-monotone (bathtub), even those that are bimodal.

Competing Interests

Authors have declared that no competing interests exist.

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