



Some Geometry of Affine Immersion of General Co-dimension

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Authors' contributions

This work was carried out in collaboration among all authors. SL designed the study, Initiated the idea after the study of affine immersion. He performed the analysis of affine immersion of general co-dimension and wrote the protocol. HEN wrote the first draft of the manuscript. GGB managed all the literature searches. All authors read and approved the final manuscript.

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Review Article

Abstract

After a careful study of some works of several authors on affine immersion of co-dimension one [1], co-dimension two [2], co-dimension three [3] and co-dimension four [4], we extend some of their fundamental equations to affine immersion of general co-dimension p . Furthermore, we extend some theorem of Frank Dillen at el in [5] to affine immersion of general co-dimension and obtain the divisibility of the cubic forms by the second fundamental forms.

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1 Introduction

There is rapid progress in research on affine differential geometry within the last two decades. Particularly on affine immersion and submersion. However, there are more to be done in these aspects. Most of the results in affine immersions are obtained on immersions of co-dimension one, two, three or four. However, some of these results can be extended to the affine immersion of general co-dimension p .

The following are some of the progress made in affine immersions; Katsumi Nomizu and Luc Vrancken in [4] investigate the geometry of nondegenerate affine surfaces in R^4 . Katsumi Nomizu and Takeshi Sasaki in [2] developed a basic machinery for centroaffine immersions of co-dimension 2 and obtain two second fundamental forms h^1 and h^2 and two cubic forms C^1 and C^2 , in which the vanishing of h^1 or h^2 is given a geometric interpretation. Tsasa Lusala in [3] studied the vanishing of the traceless part of the difference tensor field C between the Levi-Civita connections of the first and the third fundamental forms for non-degenerate surface immersions in $S^3(1)$ where a geometric meaning is given. Katsumi Nomizu and U. Pinkall in [6] showed that a non-degenerate hypersurface M with an affine structure of \mathbb{R}^{n+1} in which the cubic form C is divisible by the second fundamental form h is a hyperquadric and in [7], they proved several theorems on isometric immersions in Riemannian and Pseudo-Riemannian geometry. In [8] Hitoshi Furuhashi and Luc Vrancken investigate those immersions for which the center map is affine congruent with the original hypersurface. In terms of centroaffine geometry, they showed that such hypersurfaces provide examples of hypersurfaces with vanishing centroaffine Tchebychev operator. They also characterize them in equiaffine differential geometry using a curvature condition involving the covariant derivative of the shape operator. Katsumi Nomizu and Brian Smyth in [9] determined the holonomy groups of hypersurfaces, a generalization of the main theorem of [10] on Einstein hypersurfaces, the non-existence of a certain type of hypersurface in the complex projective space, and some results concerning the curvature of complex curves. Luc Vrancken in [11] classify the affine immersions with parallel second fundamental form in $R^{n+n(n+1)/2}$, obtaining amongst others the generalized Veronese immersions.

Some of these results are restricted to affine immersions of co-dimension one or two. In our work we extend some fundamental equations and results to affine immersion of general co-dimension p . We were motivated by the fact that these important fundamental equations are obtained in affine immersion of co-dimension one, two, three or four. That means it is possible to get these fundamental equations in affine immersion of any co-dimension p .

We organize the work as follows: At first, we give the definition of affine immersion, state the Weingarten formula and some of the fundamental equations which are relevant to affine immersion. Next, we use the definition of curvature tensor to obtain the relationship between the curvature tensor of the connection D in R^{n+p} and the induced connection ∇ in a manifold M^n when M^n is immersed in R^{n+p} and derive the equations of Gauss and Codazzi. Lastly, we extend some results in [5] and [1] to immersion of co-dimension p . We also give a necessary condition for two torsion-free affine connections to be projectively equivalent.

2 Preliminaries

In this section we introduce the notion of affine immersion of general co-dimension and obtain some important fundamental equations.

Let (M, ∇) and $(\bar{M}, \bar{\nabla})$ be differentiable manifolds with dimension n and $n + p$ respectively. Here we assume without mentioning frequently that the given affine connections have zero torsion. We give the following definition;

Definition 2.1 ([1]). . A mapping $f : (M, \nabla) \rightarrow (\bar{M}, \bar{\nabla})$ is called an affine immersion if there is a p -dimensional differentiable distribution N_x along $f : x \in M \rightarrow N_x$, a subspace of $T_{f(x)}\bar{M}$ such that

$$T_{f(x)}\bar{M} = f_*(T_x M) \oplus N_x \quad (2.1)$$

and that

$$(\bar{\nabla}_X f_* Y)_x = (f_*(\nabla_X Y))_x + (h(X, Y))_x \quad (2.2)$$

for all tangent vector fields X, Y in M , $h(X, Y) \in N_x$, at point $x \in M$ where $\nabla, \bar{\nabla}$ are the affine connections in M, \bar{M} respectively.

The distribution N_x has dimension p and is call the normal or transversal space and $h(X, Y)$ is call the second fundamental form. Since N_x is a differentiable distribution, each point $x \in M$ has a system of p differentiable vector fields $\xi_1, \xi_2, \dots, \xi_p$ called the local basis that span N_x .

Now, we let $\bar{M} = R^{n+p}$, D be the standard flat affine connection in R^{n+p} , M be an n -dimensional manifold and f an immersion of M into R^{n+p} . Then from equation (2.2), at each point $x \in M^n$ we have;

$$D_X f_* Y = f_*(\nabla_X Y) + \text{Span}\{\xi_1, \xi_2, \dots, \xi_p\} \quad (2.3)$$

This can be written as

$$D_X f_* Y = f_*(\nabla_X Y) + h^1(X, Y)\xi_1 + h^2(X, Y)\xi_2 + \dots + h^p(X, Y)\xi_p. \quad (2.4)$$

From this equation, ∇ is called the induced torsion free affine connection on M , while h^i , $i = 1, 2, \dots, p$ are symmetric tensors called the second fundamental forms. By Weingarten formula, we have the following equations;

$$\begin{aligned} D_X \xi_1 &= -S_1 X + \tau_1^1(X)\xi_1 + \tau_1^2(X)\xi_2 + \dots + \tau_1^p(X)\xi_p \\ D_X \xi_2 &= -S_2 X + \tau_2^1(X)\xi_1 + \tau_2^2(X)\xi_2 + \dots + \tau_2^p(X)\xi_p \\ &\vdots \\ D_X \xi_p &= -S_p X + \tau_p^1(X)\xi_1 + \tau_p^2(X)\xi_2 + \dots + \tau_p^p(X)\xi_p \end{aligned} \quad (2.5)$$

Here, X, Y are tangent vectors in M , S_i and τ_j^i , $i, j = 1, 2, \dots, p$ are called the shape operators and the normal connection forms respectively.

If we consider an n -dimensional manifold M together with an immersion $f : M^n \rightarrow R^{n+1}$. We call M^n a hypersurface and f is the hypersurface immersion. For each point $x \in M^n$ we choose a local field of transversal vector $\xi : x \in \mathcal{U} \leftrightarrow \xi_x$, where \mathcal{U} is neighborhood of x . In this case it means that

$$T_{f(x)}R^{n+1} = f_*(\nabla_x(M^n)) + \text{Span}(\xi_x)$$

where $\text{Span}(\xi_x)$ is the 1-dimensional subspace spanned by ξ_x . In this work we are interested in the geometry of affine immersion of general co-dimension.

Next, we obtain the relationship between the curvature tensor of the connection D in R^{n+p} and the induced connection ∇ in M .

Proposition 2.1. *Let (R, D) be the standard $n+p$ -dimensional affine space, (M, ∇) be an n -dimensional manifold and f an immersion of M into R . If R^D and R are the curvature tensors of D and*

∇ respectively. Then we have that

$$\begin{aligned}
 R^D(X, Y)Z &= R(X, Y)Z + \sum_{i=1}^p h^i(X, \nabla_Y Z)\xi_i + \sum_{i=1}^p Xh^i(Y, Z)\xi_i \\
 &- \sum_{i=1}^p h^i(Y, Z)S_i X + \sum_{i=1}^p h^i(Y, Z) \sum_{j=1}^p \tau_i^j(X)\xi_j - \sum_{i=1}^p h^i(Y, \nabla_X Z)\xi_i \\
 &- \sum_{i=1}^p Yh^i(X, Z)\xi_i + \sum_{i=1}^p h^i(X, Z)S_i Y - \sum_{i=1}^p h^i(X, Z) \sum_{j=1}^p \tau_i^j(Y)\xi_j \\
 &- \sum_{i=1}^p h^i([X, Y], Z)\xi_i,
 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
 R^D(X, Y)\xi_i &= -\nabla_X S_i Y - \sum_{j=1}^p h^j(X, S_i Y)\xi_j + \sum_{j=1}^p X\tau_i^j(Y)\xi_j - \sum_{j=1}^p \tau_i^j(Y)S_j X \\
 &+ \sum_{j=1}^p (\tau_i^j(Y) \sum_{i=1}^p \tau_j^i(X)\xi_i) + \nabla_Y S_i X + \sum_{j=1}^p h^j(Y, S_i X)\xi_j - \sum_{j=1}^p Y\tau_i^j(X)\xi_j + \sum_{j=1}^p \tau_i^j(X)S_j Y \\
 &- \sum_{j=1}^p (\tau_i^j(X) \sum_{i=1}^p \tau_j^i(Y)\xi_i) + S_i([X, Y]) - \sum_{j=1}^p \tau_i^j([X, Y])\xi_j
 \end{aligned} \tag{2.7}$$

where X, Y, Z are vector fields in M , D , ∇ are the affine connections in $R M$ respectively.

Proof. Let $\{h^1, h^2, \dots, h^p\}$ be the second fundamental forms, $\{\xi_1, \xi_2, \dots, \xi_p\}$ be transversal vectors that span the transversal distribution N_x , $\{S_1, S_2, \dots, S_1\}$ are the shape operators and $\tau_j^i, i, j = 1, 2, \dots, p$ are the normal connection forms for the affine immersion f . Then We have

$$\begin{aligned}
 D_X D_Y Z &= D_X(\nabla_Y Z + \sum_{i=1}^p h^i(Y, Z)\xi_i) \\
 &= \nabla_X \nabla_Y Z + \sum_{i=1}^p h^i(X, \nabla_Y Z)\xi_i + \sum_{i=1}^p Xh^i(Y, Z)\xi_i + \sum_{i=1}^p h^i(Y, Z)D_X \xi_i \\
 &= \nabla_X \nabla_Y Z + \sum_{i=1}^p h^i(X, \nabla_Y Z)\xi_i + \sum_{i=1}^p Xh^i(Y, Z)\xi_i \\
 &- \sum_{i=1}^p h^i(Y, Z)S_i X + \sum_{i=1}^p (h^i(Y, Z) \sum_{j=1}^p \tau_i^j(X)\xi_j).
 \end{aligned} \tag{2.8}$$

Similarly,

$$\begin{aligned}
 D_Y D_X Z &= \nabla_Y \nabla_X Z + \sum_{i=1}^p h^i(Y, \nabla_X Z)\xi_i + \sum_{i=1}^p Yh^i(X, Z)\xi_i \\
 &- \sum_{i=1}^p h^i(X, Z)S_i Y + \sum_{i=1}^p (h^i(X, Z) \sum_{j=1}^p \tau_i^j(Y)\xi_j)
 \end{aligned} \tag{2.9}$$

and

$$D_{[X, Y]} Z = \nabla_{[X, Y]} Z + \sum_{i=1}^p h^i([X, Y], Z)\xi_i \tag{2.10}$$

and so by combining equations (2.8) and (2.9) and (2.10) the curvature tensors are related by the equation

$$\begin{aligned}
 R^D(X, Y)Z &= R(X, Y)Z + \sum_{i=1}^p h^i(X, \nabla_Y Z)\xi_i + \sum_{i=1}^p Xh^i(Y, Z)\xi_i \\
 &- \sum_{i=1}^p h^i(Y, Z)S_i X + \sum_{i=1}^p h^i(Y, Z)\sum_{j=1}^p \tau_i^j(X)\xi_j - \sum_{i=1}^p h^i(Y, \nabla_X Z)\xi_i \\
 &- \sum_{i=1}^p Yh^i(X, Z)\xi_i + \sum_{i=1}^p h^i(X, Z)S_i Y - \sum_{i=1}^p h^i(X, Z)\sum_{j=1}^p \tau_i^j(Y)\xi_j \\
 &- \sum_{i=1}^p h^i([X, Y], Z)\xi_i,
 \end{aligned} \tag{2.11}$$

. Again,

$$\begin{aligned}
 D_X D_Y \xi_i &= D_X(-S_i Y + \sum_{j=1}^p \tau_i^j(Y)\xi_j) \\
 &= -\nabla_X S_i Y - \sum_{j=1}^p h^j(X, S_i Y)\xi_j + \sum_{j=1}^p X\tau_i^j(Y)\xi_j - \sum_{j=1}^p \tau_i^j(Y)S_j X \\
 &+ \sum_{j=1}^p (\tau_i^j(Y)\sum_{i=1}^p \tau_j^i(X)\xi_i),
 \end{aligned} \tag{2.12}$$

$$\begin{aligned}
 D_Y D_X \xi_i &= D_Y(-S_i X + \sum_{j=1}^p \tau_i^j(X)\xi_j) \\
 &= -\nabla_Y S_i X - \sum_{j=1}^p h^j(Y, S_i X)\xi_j + \sum_{j=1}^p Y\tau_i^j(X)\xi_j - \sum_{j=1}^p \tau_i^j(X)S_j Y \\
 &+ \sum_{j=1}^p (\tau_i^j(X)\sum_{i=1}^p \tau_j^i(Y)\xi_i)
 \end{aligned} \tag{2.13}$$

and

$$D_{[X, Y]}\xi_i = -S_i([X, Y]) + \sum_{j=1}^p \tau_i^j([X, Y])\xi_j \tag{2.14}$$

and by equations (2.12), (2.13) and (2.14) we have

$$\begin{aligned}
 R^D(X, Y)\xi_i &= -\nabla_X S_i Y - \sum_{j=1}^p h^j(X, S_i Y)\xi_j + \sum_{j=1}^p X\tau_i^j(Y)\xi_j - \sum_{j=1}^p \tau_i^j(Y)S_j X \\
 &+ \sum_{j=1}^p (\tau_i^j(Y)\sum_{i=1}^p \tau_j^i(X)\xi_i) - [-\nabla_Y S_i X - \sum_{j=1}^p h^j(Y, S_i X)\xi_j + \sum_{j=1}^p Y\tau_i^j(X)\xi_j - \sum_{j=1}^p \tau_i^j(X)S_j Y \\
 &+ \sum_{j=1}^p (\tau_i^j(X)\sum_{i=1}^p \tau_j^i(Y)\xi_i)] - [-S_i([X, Y]) + \sum_{j=1}^p \tau_i^j([X, Y])\xi_j]
 \end{aligned} \tag{2.15}$$

and so equations (2.15) and (2.11) give the results. \square

Proposition (2.1) gives the following fundamental equations.

Corollary 2.1. *Let $f : (M^n, \nabla) \rightarrow (R^{n+p}, D)$ be an affine immersion. Then the fundamental equations in affine immersion of general co-dimension are as follows:*

- a $R(X, Y)Z = \sum_{i=1}^p h^i(Y, Z)S_i X - \sum_{i=1}^p h^i(X, Z)S_i Y$
- b $\sum_{i=1}^p [(\nabla_X h^i)(Y, Z) + h^i(Y, Z) \sum_{j=1}^p \tau_i^j(X)] = \sum_{i=1}^p [(\nabla_Y h^i)(X, Z) + h^i(X, Z) \sum_{j=1}^p \tau_i^j(Y)]$
- c $(\nabla_X S_i)Y - \sum_{j=1}^p \tau_i^j(X)S_j Y = (\nabla_Y S_i)X - \sum_{j=1}^p \tau_i^j(Y)S_j X$
- d $\sum_{j=1}^p X \tau_i^j(Y) \xi_j - \sum_{j=1}^p Y \tau_i^j(X) \xi_j - \sum_{j=1}^p \tau_i^j([X, Y]) \xi_j = \sum_{j=1}^p h_i^j(X, S_i Y) - \sum_{j=1}^p h_i^j(Y, S_i X)$

Proof. Since the connection D in R^{n+p} is flat, that means $R^D = 0$. Term by term comparison in equations (2.11) and (2.15) give the desired results \square

Proposition 2.2. *Suppose that $\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_p$ are other transversal vector fields that span another transversal plane \bar{N}_x such that $\bar{\xi}_i = \lambda_{i1}\xi_1 + \lambda_{i2}\xi_2 + \dots + \lambda_{ip}\xi_p + f_*(Z_i)$ for all $i = 1, 2, \dots, p$ where Z_i are tangent vectors in M , and λ_{ij} are scalar functions. Then for all tangent vectors X, Y in M , the following equations hold:*

- 1. $\bar{\nabla}_X Y = \nabla_X Y - [h]^t A^{-1}[Z]$
- 2. $[\bar{h}] = A^{-t}[h]$.
- 3. $p\bar{\tau}_i^j(X) = \sum_{j=1}^p \rho(X)_{ij} + \sum_{j=1}^p \tau_j^j(X) \quad i = 1, 2, \dots, p$

where $A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1p} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{p1} & \lambda_{p2} & \dots & \lambda_{pp} \end{pmatrix}$ such that $\|A\| \neq 0$. $[h] = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_p \end{pmatrix}$, $[\bar{h}] = \begin{pmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \\ \vdots \\ \bar{\xi}_p \end{pmatrix}$, $[Z] = \begin{pmatrix} f_*(Z_1) \\ f_*(Z_2) \\ \vdots \\ f_*(Z_p) \end{pmatrix}$, A^t is the transpose of A , and A^{-t} is the inverse of the transpose of A .

Proof. without lost of generality, we omit f_* . Let $A = A^t$ denote the transpose of the matrix A . We know from equation (2.5) that

$$D_X Y = \nabla_X Y + h^1(X, Y)\xi_1 + h^2(X, Y)\xi_2 + \dots + h^p(X, Y)\xi_p \tag{2.16}$$

where ∇ is the induced connection with respect to the transversal vectors $\{\xi_1, \xi_2, \dots, \xi_p\}$. Again,

$$\begin{aligned} D_X Y &= \bar{\nabla}_X Y + \bar{h}^1(X, Y)\bar{\xi}_1 + \bar{h}^2(X, Y)\bar{\xi}_2 + \dots + \bar{h}^p(X, Y)\bar{\xi}_p \\ &= \bar{\nabla}_X Y + \bar{h}^1(X, Y)[\lambda_{11}\xi_1 + \dots + \lambda_{1p}\xi_p + Z_1] + \bar{h}^2(X, Y)[\lambda_{21}\xi_1 + \dots + \lambda_{2p}\xi_2 + Z_2] \\ &\quad + \dots + \bar{h}^p(X, Y)[\lambda_{p1}\xi_1 + \dots + \lambda_{pp}\xi_p + Z_p] \\ &= \bar{\nabla}_X Y + [\bar{h}^1(X, Y)\lambda_{11} + \bar{h}^2(X, Y)\lambda_{21} + \dots + \bar{h}^p(X, Y)\lambda_{p1}]\xi_1 \\ &\quad + [\bar{h}^1(X, Y)\lambda_{12} + \bar{h}^2(X, Y)\lambda_{22} + \dots + \bar{h}^p(X, Y)\lambda_{p2}]\xi_2 \\ &\quad + \dots + [\bar{h}^1(X, Y)\lambda_{1p} + \bar{h}^2(X, Y)\lambda_{2p} + \dots + \bar{h}^p(X, Y)\lambda_{pp}]\xi_p \\ &\quad + \bar{h}^1(X, Y)Z_1 + \bar{h}^2(X, Y)Z_2 + \dots + \bar{h}^p(X, Y)Z_p \end{aligned} \tag{2.17}$$

and so we have that

$$\begin{aligned}
 D_X Y &= \bar{\nabla}_X Y + [\bar{h}^1(X, Y)\lambda_{11} + \bar{h}^2(X, Y)\lambda_{21} + \dots + \bar{h}^p(X, Y)\lambda_{p1}]\xi_1 \\
 &\quad + [\bar{h}^1(X, Y)\lambda_{12} + \bar{h}^2(X, Y)\lambda_{22} + \dots + \bar{h}^p(X, Y)\lambda_{p2}]\xi_2 \\
 &\quad + \dots + [\bar{h}^1(X, Y)\lambda_{1p} + \bar{h}^2(X, Y)\lambda_{2p} + \dots + \bar{h}^p(X, Y)\lambda_{pp}]\xi_p \\
 &\quad + \bar{h}^1(X, Y)Z_1 + \bar{h}^2(X, Y)Z_2 + \dots + \bar{h}^p(X, Y)Z_p
 \end{aligned} \tag{2.18}$$

Where $\bar{\nabla}$ is the induced connection with respect to the transversal vectors $\{\bar{\xi}_1, \bar{\xi}_2, \dots, \bar{\xi}_p\}$. If we equate (2.16) to (2.18) and compare coefficient, we have in matrix form

$$\begin{pmatrix} h^1 \\ h^2 \\ \vdots \\ h^p \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{21} & \dots & \lambda_{p1} \\ \lambda_{12} & \lambda_{22} & \dots & \lambda_{p2} \\ \vdots & \vdots & \dots & \vdots \\ \lambda_{1p} & \lambda_{2p} & \dots & \lambda_{pp} \end{pmatrix} \begin{pmatrix} \bar{h}^1 \\ \bar{h}^2 \\ \vdots \\ \bar{h}^p \end{pmatrix}.$$

This is the same as $[h] = A^t[\bar{h}]$ and so $[\bar{h}] = A^{-t}[h]$. We also have the equation $\bar{\nabla}_X Y = \nabla_X Y - (\bar{h}^1 \ \bar{h}^2 \ \dots \ \bar{h}^p) \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_p \end{pmatrix}$. We can write this equation as $\bar{\nabla}_X Y = \nabla_X Y - [\bar{h}]^t[Z]$ and so we have $\bar{\nabla}_X Y = \nabla_X Y - [h]^t A^{-1}[Z]$ as required.

Similarly

$$\begin{aligned}
 D_X \bar{\xi}_i &= -\bar{S}_i(X) + \sum_{j=1}^p \bar{\tau}_i^j(X)\bar{\xi}_j \\
 &= -\bar{S}_i(X) + \sum_{j=1}^p \bar{\tau}_i^j(X)[\lambda_{j1}\xi_1 + \lambda_{j2}\xi_2 + \dots + \lambda_{jp}\xi_p + Z_j] \\
 &= -\bar{S}_i(X) + \sum_{j=1}^p [\bar{\tau}_i^j(X)\lambda_{j1}\xi_1 + \bar{\tau}_i^j(X)\lambda_{j2}\xi_2 + \dots + \bar{\tau}_i^j(X)\lambda_{jp}\xi_p + \bar{\tau}_i^j(X)Z_j]
 \end{aligned} \tag{2.19}$$

Again, we can define

$$\begin{aligned}
 D_X \bar{\xi}_i &= D_X [\lambda_{i1}\xi_1 + \lambda_{i2}\xi_2 + \dots + \lambda_{ip}\xi_p + Z_i] \\
 &= X\lambda_{i1}\xi_1 + X\lambda_{i2}\xi_2 + \dots + X\lambda_{ip}\xi_p \\
 &\quad + \lambda_{i1}D_X \xi_1 + \lambda_{i2}D_X \xi_2 + \dots + \lambda_{ip}D_X \xi_p + D_X Z_i \\
 &= X\lambda_{i1}\xi_1 + X\lambda_{i2}\xi_2 + \dots + X\lambda_{ip}\xi_p \\
 &\quad + \lambda_{i1}[-S_1(X) + \sum_{j=1}^p \tau_1^j(X)\xi_j] + \lambda_{i2}[-S_2(X) + \sum_{j=1}^p \tau_2^j(X)\xi_j] \\
 &\quad + \dots + \lambda_{ip}[-S_p(X) + \sum_{j=1}^p \tau_p^j(X)\xi_j] + \nabla_X Z_i + \sum_{j=1}^p h^j(X, Z_i)\xi_j
 \end{aligned}$$

so that

$$\begin{aligned}
 D_X \bar{\xi}_i &= -[\lambda_{i1} S_1(X) + \lambda_{i2} S_2(X) + \dots + \lambda_{ip} S_p(X)] + \nabla_X Z_i \\
 &+ X \lambda_{i1} \xi_1 + X \lambda_{i2} \xi_2 + \dots + X \lambda_{ip} \xi_p + \lambda_{i1} \sum_{j=1}^p \tau_1^j(X) \xi_j \\
 &+ \lambda_{i2} \sum_{j=1}^p \tau_2^j(X) \xi_j + \dots + \lambda_{ip} \sum_{j=1}^p \tau_p^j(X) \xi_j + \sum_{j=1}^p h^j(X, Z_i) \xi_j. \tag{2.20}
 \end{aligned}$$

If we equate (2.19) to (2.20) and compare coefficient, we obtain the following system of equations:

$$\begin{aligned}
 \bar{\tau}_i^1(X) \lambda_{i1} + \bar{\tau}_i^2(X) \lambda_{i2} + \dots + \bar{\tau}_i^p(X) \lambda_{ip} &= X \lambda_{i1} + \lambda_{i1} \tau_1^1(X) + \lambda_{i2} \tau_2^1(X) \\
 &+ \dots + \lambda_{ip} \tau_p^1(X) + h^1(X, Z_i) \\
 \bar{\tau}_i^1(X) \lambda_{i2} + \bar{\tau}_i^2(X) \lambda_{i2} + \dots + \bar{\tau}_i^p(X) \lambda_{ip} &= X \lambda_{i2} + \lambda_{i1} \tau_1^2(X) + \lambda_{i2} \tau_2^2(X) \\
 &+ \dots + \lambda_{ip} \tau_p^2(X) + h^2(X, Z_i) \\
 \bar{\tau}_i^1(X) \lambda_{ip} + \bar{\tau}_i^2(X) \lambda_{i2} + \dots + \bar{\tau}_i^p(X) \lambda_{ip} &= X \lambda_{ip} + \lambda_{i1} \tau_1^p(X) + \lambda_{i2} \tau_2^p(X) \\
 &+ \dots + \lambda_{ip} \tau_p^p(X) + h^p(X, Z_i).
 \end{aligned}$$

Finally, for each $i = 1, 2, \dots, p$, we have

$$p \bar{\tau}_i^i(X) = \sum_{j=1}^p \rho(X)_{ij} + \sum_{j=1}^p \tau_j^j(X)$$

$$\rho_{ij} = d \log \lambda_{ij} \tag{□}$$

Next, we modify one of the important theorems in centroaffine immersion of co-dimension one, of [5] to a general co-dimension p

Proposition 2.3. *Let $f : (M^n, \nabla) \rightarrow (\widetilde{M}^{n+p}, \widetilde{\nabla})$ be a non-degenerate affine immersion with transversal vector fields $\{\xi_1, \xi_2, \dots, \xi_p\}$, second fundamental forms $\{h^1, h^2, \dots, h^p\}$, transversal connection forms $\tau_j^i, i, j = 1, 2, \dots, p$ and cubic forms $C^i, i = 1, \dots, p$ where $(\widetilde{M}^{n+p}, \widetilde{\nabla})$ is a centro-affine hypersurface of R^{n+p+1} with respect to a point 0 . If we define $g : \widetilde{M}^{n+p} \subset R^{n+p+1} \rightarrow R^{n+p+1} : x \mapsto \lambda(x)x$ with $\lambda > 0$ in such a way that $g(\widetilde{M}^{n+p}) \subset R^{n+p}$ not passing through 0 . Then the following equations are satisfied;*

- i* $\bar{\nabla}_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X$
- ii* $\bar{h}^i(X, Y) = h^i(X, Y) \quad i = 1, \dots, p$
- iii* $\bar{h} = \sum_{i=1}^p \bar{h}^i(X, Y) \rho(\xi_i)$
- iv* $\sum_{j=1}^p \bar{\tau}_i^j(X) = \sum_{j=1}^p \tau_i^j(X) + \rho(X)$
- v* $\bar{S}_i(X) = S_i(X) - \rho(\xi_i)X \quad i = 1, \dots, p$

for all $i, j = 1, \dots, p$

Proof. In this case our consideration is local hence, we assume $M^n \subset \widetilde{M}^{n+p}$ and have the following immersion

- i* $(M^n, \nabla) \hookrightarrow (\widetilde{M}^{n+p}, \widetilde{\nabla})$ with induced connection ∇ , second fundamental forms h^i , shape operators S_i and transversal connection forms $\tau_j^i, i, j = 1, 2, \dots, p$
- ii* $(\widetilde{M}^{n+p}, \widetilde{\nabla}) \hookrightarrow (R^{n+p+1})$ with second fundamental form \bar{h} and transversal vector $-x$

iii $g : (M^n, \bar{\nabla}) \hookrightarrow (R^{n+p}, D)$ with induced connection $\bar{\nabla}$, second fundamental forms \bar{h}^i , shape operators \bar{S}_i , transversal vectors $\bar{\xi}_i = g_*(\xi)$ and transversal connection forms $\bar{\tau}_i^j$, $i, j = 1, 2, \dots, p$

For any vector fields $X, Y \in T_x M^n$ we have that

$$g_*(Y) = D_Y \lambda x = Y(\lambda)x + \lambda Y.$$

and so g is an immersion. Therefore,

$$\begin{aligned} D_X g_* Y &= D_X (Y(\lambda)x + \lambda Y) \\ &= Y(\lambda)X + X(Y(\lambda))x + X(\lambda)Y + \lambda D_X Y. \end{aligned} \quad (2.21)$$

But from the immersion $(M^n, \nabla) \hookrightarrow (R^{n+p}, D)$ we have,

$$D_X Y = \nabla_X Y + \sum_{i=1}^p h^i(X, Y)\xi_i + \tilde{h}(X, Y)(-x) \quad (2.22)$$

and so from equations (2.21) and (2.22) we have that

$$\begin{aligned} D_X g_* Y &= Y(\lambda)X + X(Y(\lambda))x + X(\lambda)Y + \lambda \nabla_X Y \\ &\quad + \lambda \sum_{i=1}^p h^i(X, Y)\xi_i + \lambda \tilde{h}(X, Y)(-x). \end{aligned} \quad (2.23)$$

Again, from the immersion $(M^n, \bar{\nabla}) \hookrightarrow (R^{n+p}, D)$, we have

$$D_X g_*(Y) = g_*(\bar{\nabla}_X Y) + \sum_{i=1}^p \bar{h}^i(X, Y)\bar{\xi}_i \quad (2.24)$$

But we know that $\bar{\xi}_i = g_*(\xi_i) = \xi_i(\lambda)x + \lambda \xi_i$ and so the equation (2.24) becomes

$$\begin{aligned} D_X g_*(Y) &= \lambda \bar{\nabla}_X Y + X(Y(\lambda))x + \sum_{i=1}^p \bar{h}^i(X, Y)\xi_i(\lambda)x \\ &\quad + \sum_{i=1}^p \bar{h}^i(X, Y)\lambda \xi_i \end{aligned} \quad (2.25)$$

and so by comparing coefficients in equations (2.23) and (2.25) we obtain the equations.

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{1}{\lambda} X(\lambda)Y + \frac{1}{\lambda} Y(\lambda)X, \quad (2.26)$$

$$\bar{h}^i(X, Y) = h^i(X, Y), \quad -\lambda \tilde{h}(X, Y) = -\sum_{i=1}^p \bar{h}_i(X, Y) \frac{1}{\lambda} \xi_i(\lambda). \quad (2.27)$$

We denote $\frac{1}{\lambda} X(\lambda)$, $\frac{1}{\lambda} Y(\lambda)$ and $\frac{1}{\lambda} \xi_i(\lambda)$ by $\rho(X)$, $\rho(Y)$ and $\rho(\xi_i)$ respectively where $\rho = d \log \lambda$ and we have

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + \rho(X)Y + \rho(Y)X, \quad \bar{h}^i(X, Y) = h^i(X, Y), \\ \tilde{h}(X, Y) &= \sum_{i=1}^p \bar{h}^i(X, Y)\rho(\xi_i) \end{aligned} \quad (2.28)$$

Similarly,

$$\begin{aligned} D_X g_*(\xi_i) &= D_X (\xi_i(\lambda)x + \lambda \xi_i) \\ &= \xi_i(\lambda)X + X(\xi_i(\lambda))x + X(\lambda)\xi_i + \lambda D_X \xi_i. \end{aligned} \quad (2.29)$$

But $D_X \xi_i = -S_i(X) + \sum_{j=1}^p \tau(X)_i^j \xi_j$ and so equation (2.29) becomes

$$D_X g_*(\xi_i) = \xi_i(\lambda)X + X(\xi_i(\lambda))x + X(\lambda)\xi_i - \lambda S_i(X) + \lambda \sum_{j=1}^p \tau(X)_i^j \xi_j. \quad (2.30)$$

We also have that

$$\begin{aligned} D_X g_*(\xi_i) &= -g_*(\bar{S}_i(X)) + \sum_{j=1}^p \bar{\tau}_i^j(X) g_*(\xi_j) \\ &= -\lambda \bar{S}_i(X) - (\bar{S}_i(X))(\lambda)x + \sum_{j=1}^p \bar{\tau}_i^j(X) \lambda \xi_j + \sum_{j=1}^p \bar{\tau}_i^j(X) \xi_j(\lambda)x. \end{aligned} \quad (2.31)$$

We compare coefficients in equations (2.30) and (2.31) and get the following equations;

$$\sum_{j=1}^p \bar{\tau}_i^j(X) = \sum_{j=1}^p \tau_i^j(X) + \rho(X) \quad \bar{S}_i(X) = S_i(X) - \rho(\xi_i)X \quad (2.32)$$

where $\rho(X) = d \log \lambda$. □

Next, from corollary (2.1)b, we set

$$C^i(X, Y, Z) = (\nabla_X h^i)(Y, Z) + h^i(Y, Z) \sum_{j=1}^p \tau_i^j(X). \quad (2.33)$$

It is symmetric in all the vector fields X, Y and Z in M^n . We call $C^i(X, Y, Z)$, $i = 1, 2, \dots, p$ the cubic forms of the affine immersion. For a hypersurface immersion, the cubic form is said to be divisible by the second fundamental form h if there exists a one form ρ such that, for all vector fields X, Y and Z in M^n ,

$$C(X, Y, Z) = \rho(X)h(Y, Z) + \rho(Y)h(Z, X) + \rho(Z)h(X, Y).$$

or equivalently $C(X, Y, Z) = 3\rho(X)h(X, X)$ for all $X \in T_x M^n$. We write h/C .

However, we are considering the immersion of general co-dimension and see how we can extend this notion to a co-dimension p .

Corollary 2.2. *If the cubic form C^i relative to the immersion $(M^n, \nabla) \hookrightarrow (R^{n+p}, D)$ is divisible by h^i then the cubic form \bar{C}^i relative to the immersion $g : (M^n, \bar{\nabla}) \hookrightarrow (R^{n+p}, D)$ is divisible by h^i and are related by the equation*

$$\begin{aligned} \bar{C}^i(X, Y, Z) &= C^i(X, Y, Z) - \rho(X)h^i(Y, Z) - \rho(Y)h^i(X, Z) \\ &\quad - \rho(Z)h^i(Y, X) \end{aligned} \quad (2.34)$$

for all $i, j = 1, 2, \dots, p$

Proof. From equation (2.33), the cubic form of \bar{M}^n in R^{n+p} is given as

$$\bar{C}^i(X, Y, Z) = (\bar{\nabla}_X \bar{h}^i)(Y, Z) + \bar{h}^i(Y, Z) \sum_{j=1}^p \bar{\tau}_i^j(X)$$

for $i, j = 1, 2, \dots, p$ that is,

$$\bar{C}^i(X, Y, Z) = X\bar{h}^i(Y, Z) - \bar{h}^i(\bar{\nabla}_X Y, Z) - \bar{h}^i(Y, \bar{\nabla}_X Z) + \bar{h}^i(Y, Z) \sum_{j=1}^p \bar{\tau}_i^j(X)$$

and so from theorem (2.3)

$$\begin{aligned}
 \bar{C}^i(X, Y, Z) &= Xh^i(Y, Z) - h^i(\nabla_X Y + \rho(X)Y + \rho(Y)X, Z) \\
 &\quad - h^i(Y, \nabla_X Z + \rho(X)Z + \rho(Z)X) + h^i(Y, Z)[\rho(X) + \sum_{j=1}^p \tau_i^j(X)] \\
 &= Xh^i(Y, Z) - h^i(\nabla_X Y, Z) - h^i(\rho(X)Y, Z) - h^i(\rho(Y)X, Z) \\
 &\quad - h^i(Y, \nabla_X Z) - h^i(Y, \rho(X)Z) - h^i(Y, \rho(Z)X) + h^i(Y, Z)\rho(X) + h^i(Y, Z) \sum_{j=1}^p \tau_i^j(X) \\
 &= Xh^i(Y, Z) - h^i(\nabla_X Y, Z) - \rho(X)h^i(Y, Z) - \rho(Y)h^i(X, Z) \\
 &\quad - h^i(Y, \nabla_X Z) - \rho(X)h^i(Y, Z) - \rho(Z)h^i(Y, X) + h^i(Y, Z)\rho(X) + h^i(Y, Z) \sum_{j=1}^p \tau_i^j(X) \\
 &= Xh^i(Y, Z) - h^i(\nabla_X Y, Z) - h^i(Y, \nabla_X Z) - \rho(X)h^i(Y, Z) - \rho(Y)h^i(X, Z) \\
 &\quad - \rho(Z)h^i(Y, X) + h^i(Y, Z) \sum_{j=1}^p \tau_i^j(X)
 \end{aligned}$$

Thus, we finally have that

$$\begin{aligned}
 \bar{C}^i(X, Y, Z) &= C^i(X, Y, Z) - \rho(X)h^i(Y, Z) - \rho(Y)h^i(X, Z) \\
 &\quad - \rho(Z)h^i(Y, X)
 \end{aligned} \tag{2.35}$$

and if C^i is divisible by h^i then \bar{C}^i is also divisible by h^i for all $i = 1, 2, \dots, p$ □

This gives us the same result as in co-dimension one.

Definition 2.2 ([1]). Two torsion-free affine connections ∇ and $\bar{\nabla}$ on a differentiable manifold M^n are said to be projectively equivalent if there is a 1-form ρ such that

$$\bar{\nabla}_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X$$

where $\rho = d \log \lambda$

From proposition (2.3) it is obvious that the connections ∇ and $\bar{\nabla}$ are projectively equivalent.

The function $g : M^n \rightarrow R^{n+p}$ is a projective transformation from M^n to $g(M^n) \subset R^{n+p}$. If $g(M^n)$ is a hyperplane for any suitable function λ on M^n , then a curve \bar{x} on $g(M^n)$ through a point p_0 is the intersection of $g(M^n)$ with a plane containing the line from 0 to p_0 . See [5] for details.

Theorem 2.3. Let $\bar{\nabla}$ and ∇ be two torsion free and projectively equivalent connections on a differentiable manifold M^n . Then the curvature tensor \bar{R} relative to $\bar{\nabla}$ and the curvature tensor R relative to ∇ are related by the equation

$$\begin{aligned}
 \bar{R}(X, Y, Z) &= R(X, Y)Z + [(\nabla_X \rho)Z - \rho(X)\rho(Z)]Y \\
 &\quad - [(\nabla_Y \rho)Z - \rho(Y)\rho(Z)]X
 \end{aligned} \tag{2.36}$$

where $d\rho = 0$

Proof.

$$\begin{aligned} \bar{\nabla}_X \bar{\nabla}_Y Z &= \nabla_X (\nabla_Y Z) + \rho(X) \nabla_Y Z + \rho(\nabla_Y Z) X \\ &+ X \rho(Y) Z + \rho(Y) \nabla_X Z + \rho(Y) \rho(X) Z + \rho(Y) \rho(Z) X \\ &+ X \rho(Z) Y + \rho(Z) \nabla_X Y + \rho(Z) \rho(X) Y + \rho(Z) \rho(Y) X \end{aligned} \quad (2.37)$$

$$\begin{aligned} \bar{\nabla}_Y \bar{\nabla}_X Z &= \nabla_Y (\nabla_X Z) + \rho(Y) \nabla_X Z + \rho(\nabla_X Z) Y \\ &+ Y \rho(X) Z + \rho(X) \nabla_Y Z + \rho(X) \rho(Y) Z + \rho(X) \rho(Z) Y \\ &+ Y \rho(Z) X + \rho(Z) \nabla_Y X + \rho(Z) \rho(Y) X + \rho(Z) \rho(X) Y \end{aligned} \quad (2.38)$$

$$\begin{aligned} \bar{\nabla}_{[X,Y]} Z &= \nabla_{[X,Y]} Z + \rho([X, Y]) Z + \rho(Z) [X, Y] \\ &= \nabla_{[X,Y]} Z + \rho(\nabla_X Y) Z - \rho(\nabla_Y X) Z + \rho(Z) \nabla_X Y - \rho(Z) \nabla_Y X \end{aligned} \quad (2.39)$$

We subtract 2.38 and 2.39 from 2.37, the curvature tensors of the manifolds $(\bar{M}, \bar{\nabla})$ and (M, ∇) are related as follows;

$$\begin{aligned} \bar{R}(X, Y, Z) &= R(X, Y, Z) + \rho(\nabla_Y Z) X + X \rho(Y) Z + X \rho(Z) Y + \rho(Z) \rho(Y) X \\ &- \rho(\nabla_X Z) Y - Y \rho(X) Z - Y \rho(Z) X - \rho(Z) \rho(X) Y \\ &- \rho(\nabla_X Y) Z + \rho(\nabla_Y X) Z \end{aligned} \quad (2.40)$$

and finally we have that

$$\begin{aligned} \bar{R}(X, Y, Z) &= R(X, Y, Z) + [(\nabla_X \rho) Z - \rho(X) \rho(Z)] Y \\ &- [(\nabla_Y \rho) Z - \rho(Y) \rho(Z)] X \end{aligned} \quad (2.41)$$

where $d\rho = 0$ □

Corollary 2.4. *If two affine connections $\bar{\nabla}$ and ∇ are projectively equivalent, then the curvature tensor \bar{R} with respect to $\bar{\nabla}$ is the same as the curvature tensor R with respect to ∇ iff $[(\nabla_X \rho) Z - \rho(X) \rho(Z)] Y = [(\nabla_Y \rho) Z - \rho(Y) \rho(Z)] X$*

3 Conclusions

From our work we can now conclude the following:

- a In Proposition (2.2) for a non-degenerate affine immersion, the second fundamental forms h^i , are independent of the choice of the transversal vectors ξ_i . in the affine immersion of general co-dimension p
- b It is obvious that most of the results in [5] and [1] can be extended to a more general affine immersion of co-dimension p as shown in proposition (2.3) and corollary (2.2).
- c If two affine connections $\bar{\nabla}$ and ∇ are projectively equivalent, then the curvature tensor \bar{R} with respect to $\bar{\nabla}$ is the same as the curvature tensor R with respect to ∇ if $[(\nabla_X \rho) Z - \rho(X) \rho(Z)] Y = [(\nabla_Y \rho) Z - \rho(Y) \rho(Z)] X$

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Competing Interests

Authors have declared that no competing interests exist.

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