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An Innovative Approach to the Finite Sequences of Prime Numbers

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

An innovative approach that treats prime numbers as raw experimental data making use of experimental/computational mathematics and the approximation methods is presented in order to get advanced and more exact formulations of the canonical form $P_n = P(n) \approx n \ln n$ being P_n the prime value and n its counter. The use of many different functions - such as the inverse of the modified chi-square function $1/X_k^2\,(A, n/x_o)$ with its three parameters k, A and $x_o = x_o(k)$, the function $C_\alpha n^\alpha$ with the ad-hoc α values being $k = 2 - 2\alpha$, the function $\lambda_n \ln n$, the function $\sum_h \beta_h\, ln^h n$, the harmonic series H_n and its approximation by Euler and so on - as fit functions of finite sets i.e. sequences of prime numbers leads to induction algorithms and to new relationships of the kind $P_n \approx P(n)$ though within the approximations of the calculations with all the estimations better than that of the standard formulation $P_n \approx n \ln n$. In such a manner, refined formulations with higher precisions are got showing that there are many ways to treat the finite sequences of prime numbers. Comparisons among the various methods are made in order to find the best formulation of a new and more refined relationship in a closed form that can be valid to find the most approximate value of a prime starting from its counter in the finite case.

Keywords: Prime number sequences; data fits; modified chi-square function; experimental mathematics; computational mathematics.

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1 Introduction

The issue of prime numbers in number theory has always been a challenge to face and still nowadays it remains one of the major open problems notwithstanding the many theoretical successes achieved both historically and recently [1-15] owing also to the great importance of the issue and its strict relations to physics [16-26]. There are many physical and even biological phenomena [17] which imitate the behaviour of prime numbers so that prime numbers display a paramount importance both in physics and in mathematics.

However the main concern is the fact that, unlike all the usual numerical progressions, neither an exact relationship in a closed form that links the value of a prime $P_n \in \mathrm{N}$ to its counter $n \in \mathrm{N}$ i.e. $P_n = P(n)$ has yet been found at present nor there is an analytical law that links any prime number P_n to its preceding P_{n-1}. More probably, simply none of these relations exists so that it is not possible at present to state that the induction principle holds for prime numbers.

Likewise, there are strong doubts about the nature itself of prime numbers, whether deterministic or stochastic or even showing both aspects and this dichotomy, if any, must be still investigated and explained.

Thus the prime number problem seems to be one of the so-called intractable problems [27,28]. An intractable problem is one which is very difficult to solve where, because of the great number of rules and/or ways and/or (more or less hidden) variables to be taken into account, one cannot quickly reach the goal so that there would be just a method to treat intractable problems i.e. approximations [29,30]. Many real-world problems are of this kind as for instance theories explaining the economic or the climate change are necessarily approximate due to the high number of parameters and variables involved, many of which hidden.

In mathematical domains, where a set of exact rules is available, we encounter intractability, though seldom, due to the great number of possible applications of the rules so that approximation is an attractive technique for use in problem solving because it allows to treat and solve some intractable problems and at the same time it can frequently lead to more efficient solutions to tractable problems which do not need a precise answer. As a matter of fact in many cases the exact solution is no more desirable than an approximate one.

This paper investigates how approximations can be used to produce explanations in the mathematical field of prime numbers which show a sort of hidden intractability and from this viewpoint, as for the nomenclature i.e. terminology, the term fit is used all along the article as a synonym of approximation together with the two terms data interpolation and extrapolation.

Starting from the classical Prime Number Theorem, PNT, $(x) \approx x/\ln x$, it is well known that an equivalent formulation is $P_n \approx n \ln n$. [5] and that's why in the present context the author shall refer to the standard PNT as to the approximate and asymptotic law $P_n \approx n \ln n$. Nonetheless, despite its brilliance, it is well known that this canonical representation does not work at best to get the finite value of a prime number $P_n \in \mathrm{P} \subset \mathrm{N}$ starting from its counter $n \in \mathrm{N}$ and the same happens for many other approximations such as, for instance, $P_n \approx n \cdot \{ln(n) + ln[ln(n)]\}$ and so on.

In addition another question arises. If this standard limit holds asymptotically how does the prime sequence reach this behaviour in the infinite limit? Is there any pattern on its trend towards this limit? The aim of the present work is to answer these questions too and to do so an innovative approach to the prime number problem is shown in the present report starting from some notable results got in previous studies by the same author [31-33] where the finite sequences of prime numbers have been examined from both the statistical and the analytical viewpoint fitting their differential distribution functions and the finite sequences of their frequencies $\{f_n\} \equiv \{n/P_n\}$ as well as of $\{\rho_n\} \equiv \{ln(P_n)/ln(n)\}$ by the modified chi-square function $X_k^2(A, n/x_o)$ with its three parameters k, A and $x_o = x_o(k)$ thus finding remarkable unexpected results among which the scale non-invariance [34,35] of the finite sequences of primes, their scaling laws and their correspondence with the finite progressions $\{C_\alpha \cdot n^\alpha\}$ C_α, $\alpha \in \mathrm{R}$ and $k = 2 + 2\alpha$. In addition, the implementation of the function $X_k^2(A, n/x_o)$ to the finite progressions $\{n^\alpha\}$ has led to an elementary (in that not needing the use of complex functions) and general (i.e. valid for all the zeroes from $-\infty$ up to $+\infty$) experimental evidence of the Riemann hypothesis [36].

In the present article the same innovative approach is suggested again for the values P_n/n starting from the computational viewpoint [37-48] and making use, among all the other functions, of the inverse of the modified chi-square function with k degrees of freedom

$$1 \,/\, X_k^2(A, n/x_o) \; = \; 1 \,/\, [A/(2 \cdot \Gamma_{k/2}) \cdot (n/2x_o)^{(k/2-1)} \cdot e^{-n/xo}] \qquad (1)$$

with $k \in (1.5, 2-) \subset (1.00, 2.00) \subset \mathrm{R}^+$ $\;k = 2 - 2\alpha\;$ $x_o = x_o(k) =$ decay parameter, A an ad-hoc free coefficient and the values of $\Gamma_{k/2} = \Gamma(k/2)$ easily found in the net [49, 50]. This function has been identified, together with all the other functions discussed later on, as one of the best fit functions along the whole study to match the finite sequences of $\{\mathrm{f_n}^{-1}\} \equiv \{\mathrm{P_n}/n\}$ from the analytical viewpoint, namely fitting/interpolating-extrapolating/approximating the actual data points P_n/n themselves. The aim is not only to construct a computational model of the finite sequences $\{P_n/n\}$ but even to build a new version (or new versions) of the classical formulation $P_n \approx n\, ln\,(n)$ in a closed form more reliable and precise than the old one though approximate to find the value of a prime starting from its counter $P_n = P(n)$ in the finite case.

The features of the $X_k^2[A, n/x_o(k)]$ function have been described before [31-33,36], as already told, together with all the characteristics of the fits.

The basic methodology has been to assess the best fit for a few (some hundreds or even less) actual prime numbers got from the many websites of the net [51-54] and randomly chosen belonging to a finite set/sequence, hence achieving the equation of the fitting curve, repeating the calculations for many further sets/sequences and then getting a general formula that could fit all the primes, with the due approximations, from $n = 1$ i.e. $P_1 = 2$ up to the maximum value of $n_{max} = 2 \cdot 10^{15} = 2P$ that is $P_{2P} = P_{2E15} = 75{,}674{,}484{,}987{,}354{,}031\sim \sim 75.6745E15 = 75.6745 \cdot 10^{15}$ and $ln(2P) = 35.2319235754706305696871\ldots$. In such a manner prime numbers have been treated merely just as raw experimental data, with the advantage of having neither random nor systematic errors thus simplifying the calculations a lot. However the final results have errors, though small, owing to the inaccuracies i.e. imprecisions of the fits performed and, of course, the maximum attention has been paid to reduce all these errors as much as possible that is to the least attainable values. The well-known principle of ALARA (in this case applied to errors which must be maintained As Low As Reasonably Achievable) has been kept in mind in all the calculations and fits.

In other words, starting from the actual values of n and P_n got by the net and considered for a few cases (i.e. a few data-points that is prime numbers P_n) and using the method of data interpolation and extrapolation that is fitting i.e. approximating the data by ad hoc functions, it has been possible to assess a general formula holding for all prime numbers, though with approximations that is errors or uncertainties.

In other cases, the differences or the percent differences between the actual prime values $actualP_n = P_n$ and the standard values $n \cdot ln(n)$ have been fitted by ad hoc functions f(n) thus getting $P_n \approx n \cdot ln(n) + f(n)$.

The accuracy and precision, random and systematic errors, error sources, error propagations and reliability of the results have been investigated, being these issues crucial to the whole algorithm, as explained in detail in the already cited works by the same author and as usually done in physics in treating experimental data [55-59].

The function (1) has been used as one of the best fit functions to match the finite sequences $\{f^{n-1}\} \equiv \{P_n/n\}$ and the truncated progressions $\{C_\alpha \cdot n^\alpha\}$ having domain N and co-domain R^+ with $\alpha \in (0, +1) \subset \mathrm{R}^+$. In other words any single value P_n/n can be approximated by the corresponding value of $1/X_k^2[A, n/x_o(k)]$ and of $C_\alpha \cdot n^\alpha$ with $k = 2 - 2\alpha$ thus leading to a general formula, however valid within the due approximations. The rationale underlying the entire matter has been to use this function taking advantage of the adjustment of its three parameters k, A and $x_o(k)$ which allow to optimize the fits as much as possible up to 99.99% and even more whenever possible. The same has been done also by the other fitting functions used in the study as shown later on. In other words a plot & fit algorithm has been set up. Of course all the canonical statistical markers have been calculated, examined and optimized in all the fits and in the same manner the fit parameters of any fitting process have been kept under strict control in order to assure the maximum reliability and consistency of the results.

2 New Forms of the Prime Number Theorem

Starting from the canonical form $P_n \approx n \cdot ln(n)$ [5] it is easy to check that it lacks from precision for the finite value of any prime number P_n as already shown in Fig. 1a where the comparison between the actual values P_n/n and the values of the canonical PNT i.e. $P_n/n \approx ln(n)$ is reported. The large difference between the two data sets is remarkable also in Fig. 1b looking at the percentage difference $\delta\%$ between the actual values $actP_n/n$ and the values of the canonical PNT i.e. $\delta\% = \{[({}_{\text{actual}}P_n/n) - ln(n)]/({}_{\text{actual}}P_n/n)\} \cdot 100$ (top) as well as at the difference $d_n = (actualP_n/n) - ln(n)$ (bottom). The different trends of the two variables $\delta\%$ and d_n are interesting too in that completely different one each other.

As a matter of fact, while the percentage difference $\delta\%$ approaches zero in increasing n so that the asymptotic result $\lim_{n\to\infty}\delta\% = 0^+$ is correct, yet the difference $d_n = actP_n/n - ln(n)$ increases more and more vs. $ln(n)$ being $\lim_{n\to\infty} d_n = \lim_{n\to\infty}[{}_{\text{actual}}P_n/n - ln(n)] = +\infty$

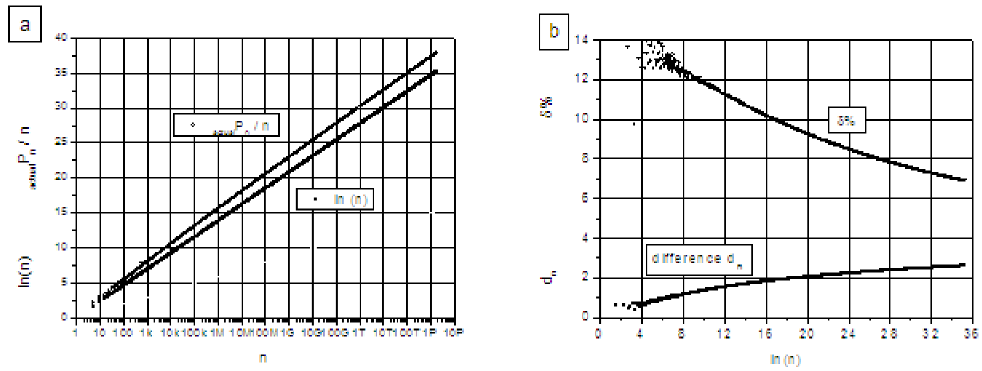


Fig. 1.a. comparison $actP_n/n$ and $ln(n)$ b. differences between $actP_n/n$ and $ln(n)$

In addition, another property can be easily verified. Just like tossing a coin [60], the initial behaviour of prime numbers P_n (or P_n/n) vs. n, as well as of the differences, shows fluctuations which seem to be random, afterwards smoothing to a well-defined curve, what suggests that, after a first transient phase, prime numbers might have a deterministic aspect which can be described analytically though approximately by means of analytical functions.

Thus, in the present report, many methods are presented which have been implemented in order to assess a relationship $P_n \approx P(n)$ more precise than the classical form $P_n \approx n \cdot ln(n)$ making use of computational mathematics and starting from the examination of a limited number of actual prime numbers P_n and afterwards extending the results to the whole set of prime numbers P.

The methodology used, that is that of experimental mathematics or computational mathematics with all the necessary approximations, is very simple in principle and it takes advantage of the recent advent of simulation based inference. In addition it needs not so many data at least in the present framework. Just for example, the famous historic formula

$$\sum_{k=0\to n} k = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n = n \cdot (n+1)/2$$

got by Gauss considering all the $(n/2)$ identical and symmetrical sums

$$1 + (n) = 2 + (n-1) = 3 + (n-2) = \ldots = (n-2) + 3 = (n-1) + 2 = (n) + 1$$

for $\forall\, n \in N$

can be found also using computational mathematics starting from few data, just the initial ones, and applying the induction principle considering that:

$$\sum_{k=0\to 1} k = 1 = 1 \cdot 1.0 = 1 \cdot 2/2 = 1 \cdot (1/2 + 1/2)$$
$$\sum_{k=0\to 2} k = 3 = 2 \cdot 1.5 = 2 \cdot 3/2 = 2 \cdot (2/2 + 1/2)$$
$$\sum_{k=0\to 3} k = 6 = 3 \cdot 2.0 = 3 \cdot 4/2 = 3 \cdot (3/2 + 1/2)$$
$$\sum_{k=0\to 4} k = 10 = 4 \cdot 2.5 = 4 \cdot 5/2 = 4 \cdot (4/2 + 1/2)$$
$$\sum_{k=0\to 5} k = 15 = 5 \cdot 3.0 = 5 \cdot 6/2 = 5 \cdot (5/2 + 1/2)$$
$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$
$$\sum_{k=0\to n} k = \ldots\ldots.\ldots\ldots.. = n \cdot (n+1)/2 = n \cdot (n/2 + 1/2)$$

Of course both methods lead to the same result with the difference that the latter makes use of computational mathematics, of the interpolation and the extrapolation principles, as called in physics for the treatment of experimental data points, equivalent to the induction principle in mathematics.

As another example, it has to be considered that the well-known standard prime number theorem in its original form was conjectured by Gauss (again) just on the basis of the behaviour of the first thousand primes and confirmed theoretically only later on also with corrections $O(\sqrt{n})$, $O(\sqrt{n} \cdot ln(n))$, etc.

These examples are reported just to show the power of experimental mathematics and its methodology. In addition, in the present case, the technique of treating experimental data (that is prime numbers) has been implemented with all its basic concepts of probability, statistics, distribution of errors with their propagations, correlations and so on as already pointed out.

2.1 The canonical PNT corrected by the exponential decay or growth

It has been already told that two main features are evident in the previous Fig. 1a and b: the first is the high fluctuations of the data points at low values of n, basically up to $ln(1{,}000) \div ln(10{,}000) \sim 6.90775527898 \div 9.2103403719$ which seem to be randomly spread; the second is the regular behaviour of the values starting from approximately these values.

Therefore, ignoring the initial terms, the exploitation of these two circumstances leads to fit the $\delta\%$ values, thus the corresponding data points, after dropping the first ones by an ad-hoc function, described by an ad-hoc curve in the plane, in this case chosen as an exponential decay of $\delta\%$ vs. $ln(n)$ as in Fig. 2a leading to

$$\delta\% \approx (4.184 \pm 0.012) + (11.6907 \pm 0.0055) \cdot n^{-1/(24.0337\pm 0.06184)} \quad R^2 = 0.99995 \quad X^2_{t\text{-}v} = 9.825E-5$$

with the values of the two fit markers R^2 (very close to 1.) and $X^2_{tv} = X^2_{test-value}$ (very low) and the errors on the coefficients (they too very low ranging from 0.5‰ to 3‰) assuring the goodness of the fit itself. Hence, neglecting the uncertainties in order to simplify the calculations, one gets

$$P_n/n \approx ln(n) + \delta\% \approx ln(n) \cdot \{1 + [4.184 + 11.6907 \cdot n^{-1/24.0337}]/100\}$$

the value $to = 24.0337$ being the decay constant of the exponential function. The further percentage difference $\delta\delta\%$ between the actual P_n/n values and the values got by this latter best fit formula are calculated and shown in the next Fig. 2b for about 1,000 values of

$$ln(n) = ln(2) \to ln(2E15) = 0.69314718055994530 \dots \to 35.231923575470630569$$

Again, after some initial fluctuations inessential for the calculations and the viewpoint here adopted, one can do a further fit, again by an exponential decay curve, for about 800 values of $\delta\delta\%$ showing (Fig. 2b) a regular trend from $ln(n) \sim 12.$ up to $ln(n) \sim 35.231923$ i.e. $n \sim 162{,}755 \to 2E15$ that is $Pn = 2{,}201{,}281 \to 75{,}674{,}484{,}987{,}354{,}031$ leading to the formula

$$\delta\delta\% \approx (0.2283 \pm 0.0014) + (2.346 \pm 0.004) \cdot n^{-1/(14.22232 \pm 0.04028)} \quad R^2 = 0.99976$$

$${}_{tv}X^2 = 1.29259E-5$$

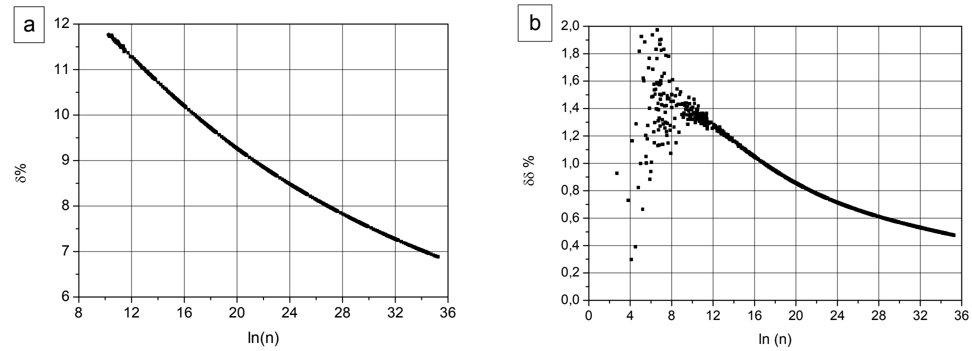


Fig. 2. a. Values of $\delta\%$ and their exponential decay fit b. The related $\delta\delta\%$ difference

that is again an exponential decay with decay constant $t_o = 14.22232$.

Again neglecting the uncertainties just to simplify the calculations, the final result is

$$P_n/n \approx ln(n) + \delta\% + \delta\delta\% \approx ln(n) + ln(n) \cdot \{[4.184 + 11.6907 \cdot n^{-1/24.0337}]/100 + [0.2283 + +2.346 \cdot n^{-1/14.22232}]\}/100$$

Once again the newly found difference $\delta 3\% = \delta\delta\delta\%$ with the classic formula $P_n = n \cdot ln(n)$ is calculated vs. ln(n) (though not shown) displaying a curve looking like the superposition of a decay curve of exponential type (thus a recurrence effect) and maybe damped oscillations.

At this point the fitting procedure stops in that it is not easy to find the fit function. One of the interesting effects is that the initial random fluctuations appear later and later as the k^{th} order of $\delta_k\%$ increases and that the values of $\delta_k\%$ diminish more and more vs. the order k (see the scale in the Figs. 2a and b so that the method appears to be promising.

Plain to say of course that the procedure might go on thus showing that the best final formulation would seem to be the summation

$$P_n/n \approx ln(n) \cdot [1 + (\alpha_o \pm \delta\alpha_o) + \sum_{k=1 \to N} (\alpha_k \pm \delta\alpha_k) \cdot n^{-1/(\tau k \pm \delta\tau k)}]$$

with the many coefficients $(\alpha_k \pm \delta\alpha_k)$ and the many decay constants $(\tau_k \pm \delta\tau_k)$ to be assessed up to the maximum attainable precision so that at least a mainframe, instead of the simple PC used by the author, would be necessary for a thorough and deep investigation of the problem. However the procedure could stop already with Fig. 2b as it clearly shows that $\lim_{n\to\infty} \delta\delta\% \sim 0^+$ and no further approximation should be necessary.

Nonetheless what is important, in the present context, is to set down a process useful to help in solving, or at least showing the way to solve, the problem of prime numbers and of their apparent unpredictability and volatility.

Turning back to the simple difference $d_n = actP_n/n - ln(n)$ (the bottom curve in Fig. 1b) it can be fitted as in the next Fig. 3a by a double exponential growth dropping the initial data i.e.

$$d_n = {}_{actual}P_n/n - ln(n) \approx -(0.23989 \pm 0.0145) + (1.54092 \pm 0.0504) \cdot [1 - n^{-1/(7.81528\pm0.26674)}] + +(2.61733 \pm 0.0507) \cdot [1 - n^{-1/(50.12546\pm4.69417)}] \qquad R^2 = 0.999968 \qquad {}_{test\text{-}value}X^2 = 4.6E-6$$

and again plotting the further differences with the canonical PNT vs. $ln(n)$ one gets the Fig. 3b thus obtaining a very good approximation not only for the values themselves ($\approx 10^{-4} \div 10^{-5}$) but also for the trend probably that of damped fluctuations around zero.

Plain to say that the values of the different statistical markers used for the different fits, i.e. the Bravais-Pearson correlation coefficient R^2, the non-linear index of correlation I, the least square sum LSS, the chi-square test value tvX^2, the standard deviation $\sigma^2 = SD$ (the variance $\sigma = \sqrt{SD}$is used too) etc. are typical of the fit and are a measurement of the goodness of the fit itself.

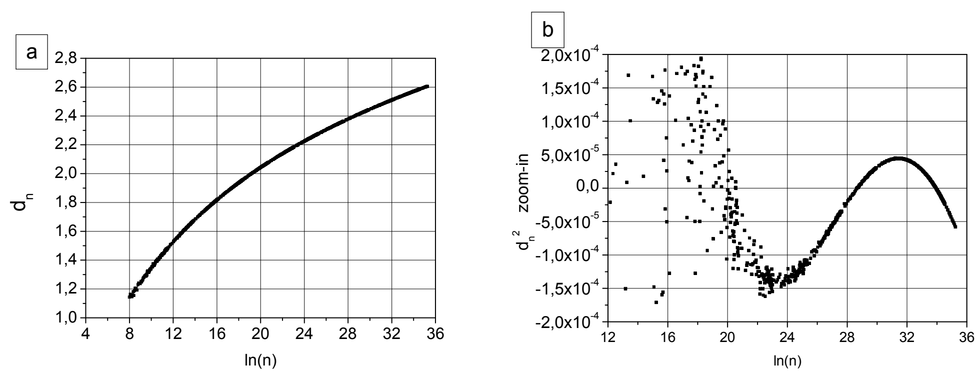


Fig. 3. a. Fit of d_n by a double exponential growth. B. The second difference $dd_n = d_n^2$

The same consideration can be applied to the fitting function that not only can be of many kinds but it can be even expressed in many ways as for instance by the difference $d_n == actP_n/n - ln(n)$ in terms of $ln[ln(n)]$ leading to a linear fit vs. $ln[ln(n)]$. The use of this correlation function leads to a PNT of the form (with negligible errors i.e. inaccuracies)

$$P_n/n \approx ln(n) + d_n \approx ln(n) - 0.95925 + 1.00147 \cdot ln[ln(n)]$$

a more accurate formulation of the canonical form, already known theoretically.

Going on with this procedure the second difference $dd_n = d_n^2$ with the actual values of P_n/n can be got though the comparison between the related plot shows that the result is worse than in the previous case.

2.2 The refinement of the canonical PNT by a polynomial

Starting again from the canonical form $P_n/n \approx ln(n)$ an obvious thing to do is to fit the actual values of P_n/n by a polynomial in ln(n) of the first, second, third and so on degree i.e.

$$\frac{P_n}{n} \approx \alpha_0 + \alpha_1 \cdot ln(n) + \alpha_2 \cdot ln^2(n) + \alpha_3 \cdot ln^3(n)\ \dots +\ \alpha_k \cdot ln^k(n) =\ \alpha_0 + \sum\nolimits_{h=1\to k} \alpha_h \cdot ln^h(n)$$

he study has been limited to the value $k = 5$ and for any of these cases the linked fit has been examined with the results shown hereafter, where $N = 100$ is the number of the data points, being $p < 1E\text{–}4$the probability that a data point may fall off the fit curve between $\pm\sigma$.

1[rst] degree fit $P^n/n\ \approx\ (0.764 \pm 0.027) + (1.059 \pm 0.001) \cdot ln(n) \quad R^2 = 0.99992 \quad \sigma = 0.109$

2[nd] degree fit $P_n/n\ \approx\ (0.28848 \pm 0.01352)\ +\ (1.11946 \pm 0.00155) \cdot ln(n) - (0.00157 \pm\ \pm 3.9E\text{–}5) \cdot ln^2(n) \quad R^2 = 0.99998 \quad \sigma = 0.02537$

3[rd] degree fit $P_n/n\ \approx\ (0.07212 \pm 0.01259)\ +\ (1.16378 \pm 0.00237) \cdot ln(n) - (0.00411 \pm \pm 1.312E\text{–}4) \cdot ln^2(n)\ +\ (4.28116 \pm 0.219539)E\text{–}5 \cdot ln^3(n) \quad R^2 = 1.000000 \quad \sigma = 0.01116$

4[th] degree fit $P_n/n \approx (0.06734 \pm 0.02643) + (1.165144 \pm 0.00702) \cdot ln(n)\text{–}(0.00424 \pm\ \pm 6.253E\text{–}4) \cdot ln^2(n) +\ (4.7442 \pm 2.2597)E\text{–}5 \cdot ln^3(n)\text{–}(5.8543 \pm 0.28433)E\text{–}8 \cdot ln^4(n) \quad R^2 = 1.000000 \quad \sigma = 0.01122$

5[th] degree fit $P_n/n\ \approx\ (0.25167 \pm\ 0.04979)\ +\ (1.09778 \pm 0.01715) \cdot ln(n) - (0.00451 \pm \pm 0.00214)\ \cdot ln^2(n) - (4.66717 \pm 1.2308)E\text{–}4 \cdot ln^3(n)\ +\ (1.38534\ \pm\ 0.329314)E\text{–}5 \cdot ln^4(n) - (1.40693 \pm\ \pm 0.331994)E\text{–}7 \cdot ln^5(n) \quad R^2 = 1.000000 \quad \sigma = 0.01028$

Of course the value $R^2\ =\ 1.000000$ means merely that the precision of the algorithm used for the fit is limited to the value of 10^{-6} = 1E–6 that is up to the 6[th] decimal digit. All these five curves plotted on a P_n/n vs. $ln(n)$ graph (though not shown) are undistinguishable one from each other so that, in order to clarify the whole situation, the five plots of the percentage differences $\delta_{\%} = [(actP_n/n\text{–}FIT)/actP_n/n] \cdot 100$ are shown in the next Fig. 4a (for the 1[rst] and 2[nd] degree polynomial fits) and b (for the 3[rd], 4[th] and 5[th] degree polynomial fits).

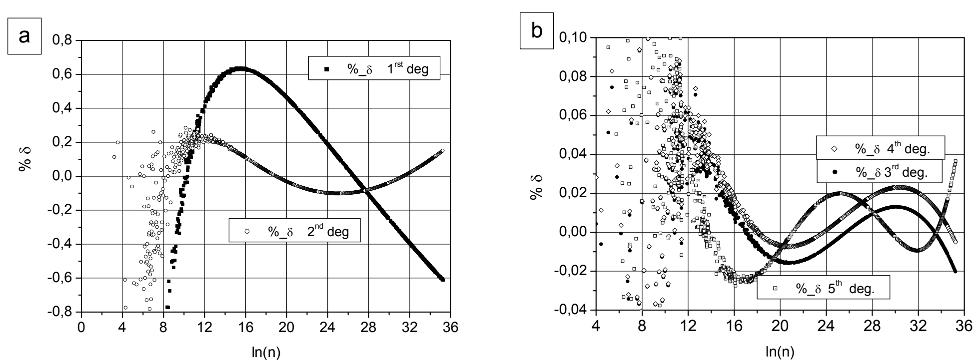


Fig. 4. A. The 1[rst] & 2[nd] degree % difference b. The 3[rd], 4[th] & 5[th] degree % difference

It is manifest that a polynomial fit of higher and higher degree is a good choice, in that the error term diminishes more and more despite the fact that the 3rd and 4th degree fits appear similar. Therefore higher degree polynomials would be very useful though difficult to manage and other solutions in a closed form have been looked for and found.

The next step is quite plain and consists in fitting the actual values of P_n/n by the polynomials

$$P_n/n \; \approx \; \sum_{m=0\to M} \alpha_m \cdot ln^{-m}(n)$$

where the value of $M = 2$ has been chosen just as an example. Thus the differences

$$d_n \; = actP_n/n - \sum_{m=0\to 2} \alpha_m \cdot ln^{-m}(n)$$

have been reported vs. $ln^{-1}(n)$ as shown in the next Fig. 5a and fitted by the relationship

$$d_n \approx (0.812 \pm 0.004) \; + \; (2.868 \pm 0.002) \cdot e^{-1/(0.05935 \pm 2.27E-4)} \quad R^2 = 0.9996 \quad tvX^2 = 6.8E-5$$

thus obtaining a formula, though approximate, for a better PNT that is (apart from the errors)

$$P_n/n \; \approx \; ln(n) + d_n \; \approx \; ln(n) + 0.812 + \; 2.868 \cdot n^{-1/0.05935}$$

Also this formula has been compared with the actual values P_n/n finding that the trend of the further difference dd_n is that reported in Fig. 5b where, again as in the other cases already found, after some initial random fluctuations the trend becomes regular.

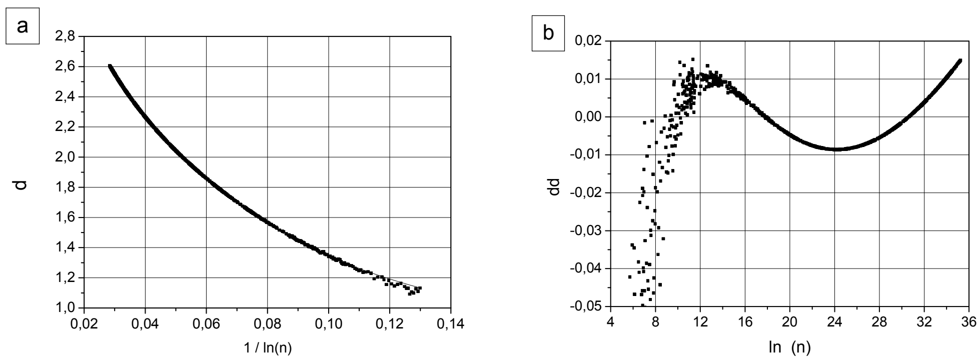


Fig. 5. a. $d_n = actP_n/n\text{–}fit$ vs. $1/\ln(n)$ b. 2nd differences $dd_n = actP_n/n\text{–}fit$ vs. $ln(n)$

Of course other approaches might be attempted as for instance reporting the actual values $ln(P_n/n)$ vs. the variable $ln[ln^{-1}(n)]$ as shown in Fig. 6a. to get the (weakly) quadratic fit

$$ln(P_n/n) \approx 0.19236 \pm 9.254E-4 - (0.9815 \pm 6.E-4) \cdot ln[ln^{-1}(n)] - (0.0044 \pm 1.E-4) \cdot \cdot$$
$$ln2[ln-1(n)] \qquad R2 = 1.000000 \qquad \sigma = SD = 4.732E-4 \qquad p < 1E-4$$

and afterwards calculating the differences with the actual values of P_n/n as in Fig. 6b that shows that the difference d in this case is fitted by

$$d_n \; \approx - \; 0.99684 \pm 0.00554 - (1.02763 \pm 0.00383) \cdot ln[ln^{-1}(n)] - \left(0.00446 \pm 6.516E-4\right) \cdot$$
$$ln^2[ln^{-1}(n)] \qquad R^2 \; = \; 0.99995 \qquad \sigma \; = \; 0.00283 \qquad p \; < \; 1E-4$$

being p the probability that a value may fall off the fit between $\pm\sigma$, with the limit $\lim_{n\to\infty} d_n = 0^+$ The latest figure is to be compared with the analogue Figs. of the differences of the preceding techniques.

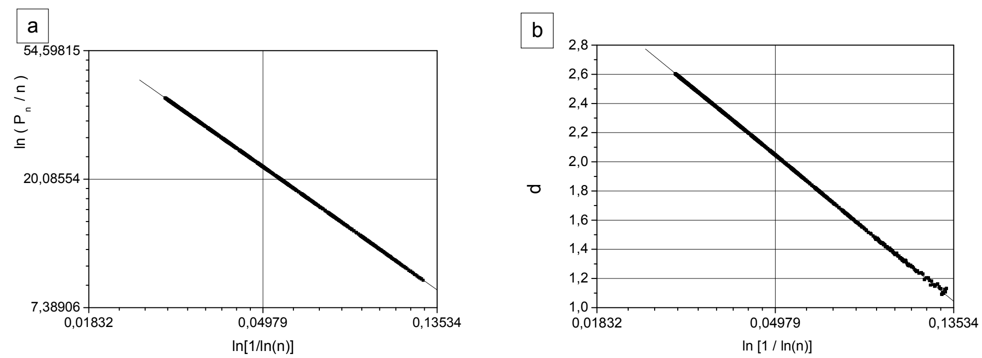


Fig. 6. a. Quadratic fit in $\boldsymbol{ln[ln^{-1}(n)]}$ b. Difference d between $\boldsymbol{actP_n/n}$ and fit

As in the earlier situations the fit procedure stops here for space reasons. However there are clues that one is entering a new area of prime numbers that is the stochastic area. This is not the context in which to debate such a matter owing to its deepness and vastness, nonetheless it is clear that this issue, together with many other topics emerging from the present study and not yet examined, is of the utmost interest to understand the inner nature of prime numbers and it deserves future profound investigations. For instance it would be very interesting to examine the statistical behaviour of the distances between any actual prime number and the fit curve got by any technique in order to ascertain whether or not there is a stochastic trend in addition to the deterministic one found just now in this study. This will be the matter of future studies.

Once again, one of the most important remarks is that the whole matter and the investigation methodology adopted is well suited to many analyses and versions thus showing a high versatility that can be used in future in view of the utmost results. As a matter of fact, the power of the methodology of computational mathematics and the ensuing approximations is the paramount statement of the entire research, well beyond any single result found.

2.3 The refinement of the canonical method by $\boldsymbol{\lambda_n \cdot ln(n) = \lambda(n) \cdot ln(n)}$

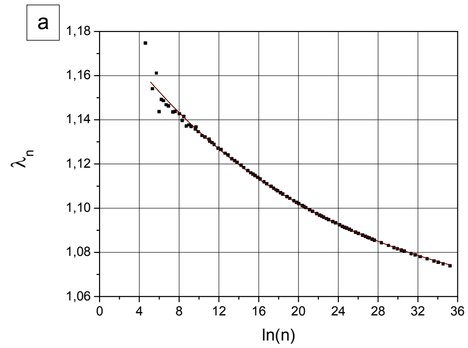
A further technique has been using an ad hoc coefficient $\lambda_n = \lambda(n)$ so that a more refined relation of the type $P_n/n \approx \lambda(n) \cdot ln(n) = \lambda_n \cdot ln(n)$ can be written. The difference with the $P_n/n = \alpha_0 + \alpha_1 \cdot ln(n)$ method is evident in that now the coefficient α_1 i.e. $\lambda_n = \lambda(n)$ is not fixed but it depends on the prime counter n.

The best fit of few primes (about 100 values from $n \sim 10{,}000$ up to $n = 2E15 = 2 \cdot 10^{15}$, neglecting the few first values) leads to the function

$$\lambda_n = \lambda(n) \approx 1.000 + (0.13894 \pm 0.00206) \cdot n^{-1/(21.08258 \pm 1.12645} + (0.0482 \pm 0.0032) \cdot$$
$$\cdot\, n^{1/3.4359E127} \qquad X^2_{tv} = 4.479E{-}6 \qquad R^2 = 0.9921$$

thus

$$P_n \approx \lambda_n \cdot n \cdot ln(n) \approx n \cdot ln(n) \cdot [1.000 + 0.13894 \cdot n^{-1/21.08258}]$$



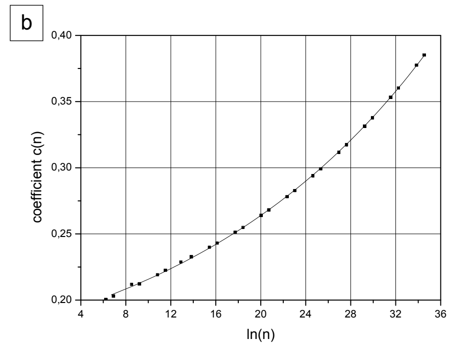


Fig. 7. a. The coefficient $\lambda_n = \lambda(n)$ vs. $ln(n)$ b. The trend of the coefficient $c(n)$

where the latter form neglects both the term $n^{1/3.4359E127} \sim 1.00$ and the errors (i.e. uncertainties) on the parameter values.

One can easily check (from Fig. 7a too as well as analytically) that the asymptotic trend of the canonical PNT is respected at all being $\lim_{n \to \infty} \lambda_n = 1$.

However the result, though apparently better than the classical $P_n \approx n \cdot ln(n)$, is not satisfying first of all because the error (approximately 4.3% though not shown) does not decrease vs. n but on the contrary it seems to increase so that some corrections must be brought. As a matter of fact, instead of the fixed coefficient (0.13894 ± 0.00206) one should write the varying coefficient $c_n = c(n)$ owing to the scale non-invariance of the finite sequences of prime numbers, as shown later on. Thus

$$\lambda_n = \lambda(n) \approx 1.0000 + c(n) \cdot n^{-1/(21.08258 \pm 1.12645)}$$

and a deep investigation shows that

$$c(n) \approx (0.1281 \pm 0.0026) + (0.057 \pm 0.002) \cdot e^{+\ln(n)/(22.83 \pm 0.41} \quad X^2 = 6.58E\text{-}7 \quad R^2 = 0.99981$$

Hence the final formula for a PNT refined in such a way is

$$P_n \approx n \cdot \lambda(n) \cdot ln(n) \approx n \cdot ln(n) \cdot [1 + (0.12812 + 0.05656 \cdot n + 1/22.83) \cdot n^{-1/21.08258}]$$

neglecting the errors just for practical calculations.

The Fig. 7b describes the trend of the coefficient c(n) while the Fig. 8 shows the inaccuracy of this technique that is the % error of $P_n/n \approx \lambda(n) \cdot ln(n)$ much better than that of the previous technique.

At the present time this latest method appears to be very good in that leading to very small errors as in Fig. 8 a & b the trend of which again seems to suggest that of damped oscillations around zero, though still to be thoroughly checked with many more data.

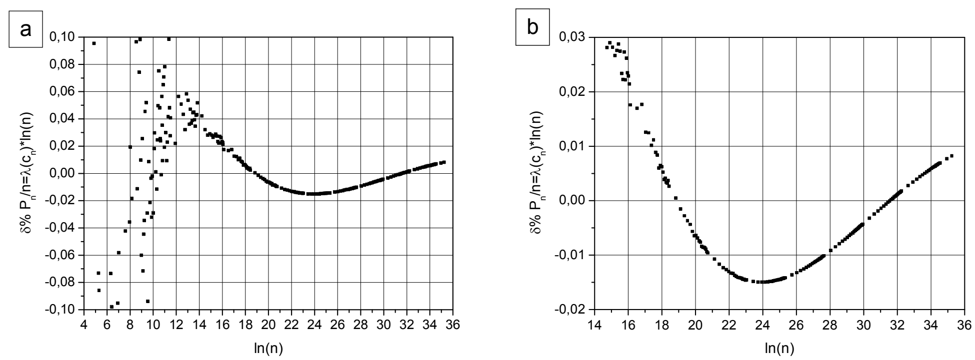


Fig. 8. a. The percent error of $\lambda(n)\cdot ln(n)$ vs. $ln(n)$ b. Zoom of the previous plot 8a

As a matter of fact it is plain and evident that many more data points, namely primes, are required in order to validate this latest assumption not only in this case but also in all the other cases showing such apparent trend.

2.4 The Fit of P_n/n by the modified chi-square function

An important fact concerning the finite sequences of prime numbers as already shown in other works by the same author [31-33] is to be mentioned. This circumstance is the leading point i.e. the core of the whole study in that it can be shown that any finite sequence of $\{P_n/n\}$ can be fitted by the functions $1/X_k^2(A, n/x_o)$ and $C_\alpha \cdot n^\alpha$ at the utmost level with $k = 2 - 2\alpha$ within the ranges $\alpha \in (0, +1)$ and $k \in (0, +2)$. In addition, the function $\lambda_n \cdot ln(n)$ too is a fit function of these sequences as already shown. The next Fig. 9 a and b shows the three fits of the actual values of $\{P_n/n\}$ by $\lambda_n \cdot ln(n)$ by $1/X_k^2(A, n/x_o)$and by $C_\alpha \cdot n^\alpha$ together with the canonical form $P_n/n \approx \lambda(n)$ in the example of the finite sequence of 200 data-points of $\{P_n/n\}$ $n = 1K \to 10G = 1\cdot 10^3 \to 1\cdot 10^{10}$ and the $\delta\%$ differences between the fits and the actual P_n/n values. The features are: $P_{10G}/10G$ fit

by $\lambda \cdot \ln(n) \approx 1.097050419011610 \cdot \ln(n)$ $R^2 = 0.99999855$ $I = 0.997737$

$X_{tv}^2 = 0.020084$ $LSS = 0.205178$

by $C_\alpha \cdot n^\alpha \approx 8.97848152322202 \cdot n^{0.0449075421740}$ $R^2 = 0.999420$ $I = 0.998835$

$X_{tv}^2 = 0.010509$ $LSS = 0.115539$

by $1/X_k^2(A, n/x_o)$ with $A = 1E{-}3$ $x_o = 1.76769784891350E+52$

$\Gamma_{k/2} = 1.028002319483930$ $k = 2 - 2\alpha = 2 - 2 \cdot 0.0449075421740 = 1.9101849156520$

$R^2 = 0.999420$ $I = 0.998835$ $X_{tv}^2 = 0.010456$ $LSS = 0.115539$

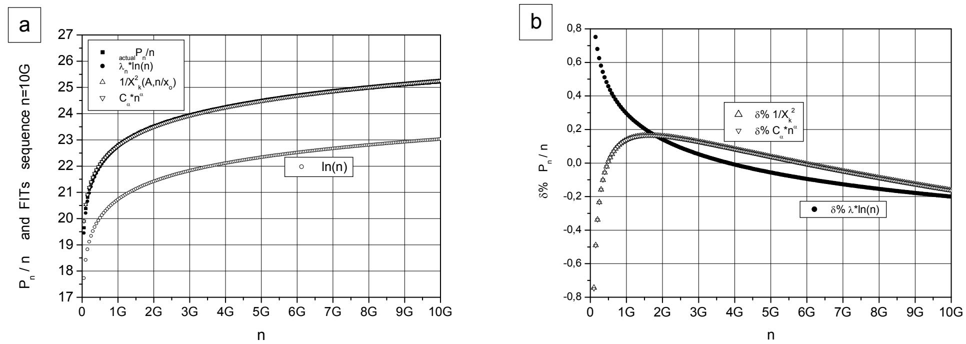


Fig. 9. a. Fits by $\lambda \cdot \ln(n)$, $1/X_k^2$,$C_\alpha \cdot n^\alpha$ & $ln(n)$ b. The δ% differences for the 3 previous fits

The next Fig. 10 a and b (semi-log plots) show the case of the finite sequence of 200 data-points of $\{P_n/n\}$ up to $40T = 40E12$ showing the following features: $\{P_{40T}/40T\}$ fit

by $\lambda \cdot \ln(n) \approx 1.080994362779060 \cdot \ln(n)$ $R^2 = 0.999996$ $I = 0.997890$

$X_{tv}^2 = 0.013165$ $LSS = 0.19187$

by $C_\alpha \cdot n^\alpha \approx 12.2869934353820 \cdot n^{0.0323447413730}$ $R^2 = 0.999705$ $I = 0.999409$

$X_{tv}^2 = 3.728E\text{–}3$ $LSS = 0.05871$

by $1/X_k^2(A, n/x_o)$ with $A = 1E\text{–}3$ $x_o = 2.17161641280590E + 68$

$\Gamma_{k/2} = 1.019736442394910$ $k = 2\text{–}2\alpha = 2\text{–}2 \cdot 0.032344741313730 = 1.93531051737254$

$R^2 = 0.999705$ $I = 0.999409$ $X_{tv}^2 = 3.7184E\text{–}3$ $LSS = 0.05871$

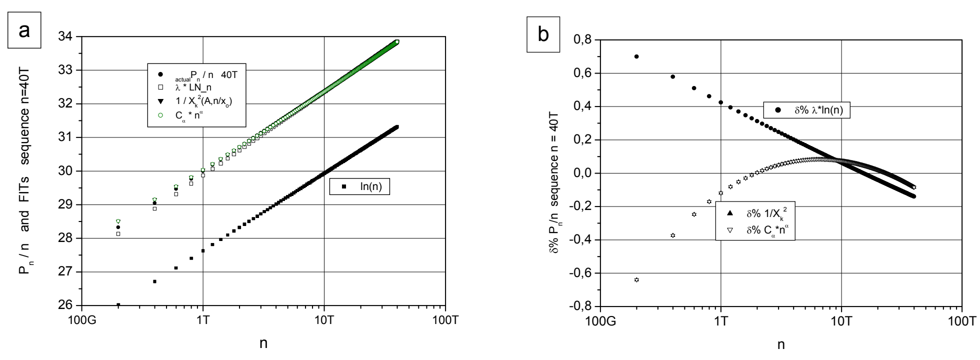


Fig. 10. a. Fits by $\lambda \cdot \ln(n)$, $1/X_k^2$,$C_\alpha \cdot n^\alpha$ & $ln(n)$ b. The % differences for the 1rst three fits

where two features are evident: the improvement of all the fit markers in increasing the value of n i.e. of P_n and the almost perfect matching between the fit markers (R^2, I, X_{tv}^2 & LSS) of the $1/X_k^2(A, n/x_o)$ fit and the $C_\alpha \cdot n^\alpha$ fit. Actually in all the cases examined the fits between these two latest functions have the following features: $R^2 = 1.000 \ldots$ up to the 15th decimal digit; $I = 1.000 \ldots$ up to the 15th decimal digit; $LSS \sim (10^{-28} \div 10^{-26})$ and $X_{tv}^2 \sim (10^{-29} \div 10^{-27})$.

As a matter of fact, being $1/X - k_k^2(A, n/x_o) = 1/[A/(2 \cdot \Gamma_{k/2}) \cdot (n/2x_o)^{(\frac{k}{2}-1)} \cdot e - n^{-n/2xo}] \approx\approx C_\alpha \cdot n^\alpha$ it is enough to set $C_\alpha = C_\alpha(k, x_o) = 1/[A/(2 \cdot \Gamma_{k/2}) \cdot (1/2x_o)^{(\frac{k}{2}-1)}]$ to get the result $n^{1-k/2} \cdot e^{-n/2xo} \approx n^\alpha$ and being $ln(x_o) >> ln(n)$ always as shown in Fig. 11a, i.e. $x_o >> n$ so that $e^{-n/2xo} \sim 1$ one gets $n^{1-k/2} = n^\alpha$ i.e. $k = 2 - 2\alpha$ at the utmost precision.

Another interesting feature of these fits is that one has the possibility to choose the fitting functions not only on linear plots (Fig. 9 a & b) but also on semi-log plots as shown in Fig. 10 a & b thus being able to select the best way to illustrate a fit.

In both cases (the linear plot and the semi-log one) the huge difference between the actual values of P_n/n and the classic $P_n/n \approx ln(n)$ is remarkable. In addition, being in these two examples $\lambda_{10G} = \lambda(10G) = 1.09705041901161 \neq \lambda_{40T} = \lambda(40T) = 1.08099436277906$ one has the validation of the dependence of λ_n on n i.e. $\lambda_n = \lambda(n)$ and of the soundness of the previous fit by $\lambda n_n \cdot ln(n)$ as well as of the scale non-invariance of the finite sequences of prime numbers.

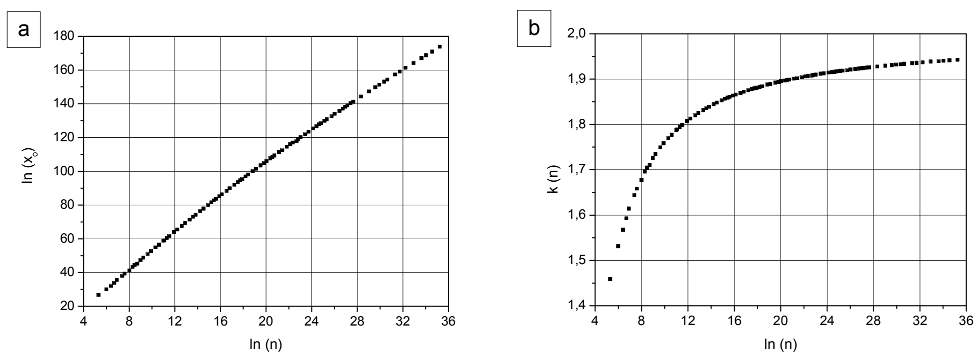


Fig. 11. A. $ln(x_o)$ vs. n for the 90 $\{P_n/n\}$ sequences b. $k(n)$ for the 90 $\{P_n/n\}$ sequences

Many interesting results and findings have been obtained in such a manner, among which the trend of all the parameters vs. n, that is $x_o = x_o(n)$ in the previous Fig. 11a with $\lim_{n\to\infty} n/x_o = 0^+$ $k = k(n)$ as in the previous Fig. 11b with $\lim_{n\to\infty} k(n) = 2^-$ and also $\Gamma_{k/2} = \Gamma_{k/2}(n)$ in the next Fig. 12a with $\lim_{\square\to\infty} \square_{\square/2} = 1^+$ useful to derive reliable formulations for the relationship $P_n \approx P(n)$. The matter has been also deeply treated in previous works, already cited, by the same author for the variables $\{\rho_n\} \equiv \{ln(P_n)/ln(n)\}$ and $\{f_n\} \equiv \{n/P_n\}$ with remarkable findings.

The fact that it is possible to treat the fits in two different ways - remarking that the previous graph in Fig. 9a can be plotted vs. n also on a log scale appearing like Fig. 10a (and the same for the uncertainties Figs. 9b and 10b) - shows that there are several ways to make the fits of the finite sequences of P_n/n and of P_n themselves too.

Another remarkable finding of this study is the result that the finite sequences of prime numbers have not the property of scale invariance and that scale laws hold for them. In addition any prime sequence $\{P_n/n\}$ is in correspondence with one and only one progression $\{C_\alpha n^\alpha\}$ as well as with the function $\lambda_n \cdot ln(n)$ for which the limit exists $\lim_{n\to\infty} n/ = 1^+$ thus giving back the standard law $P_n \approx n \cdot ln(n)$. So it is evident that useful relationships can be got for the prime sequences $\{P_n/n\}$ in order to obtain a relationship for the values of $P_n = P(n)$, yet approximate.

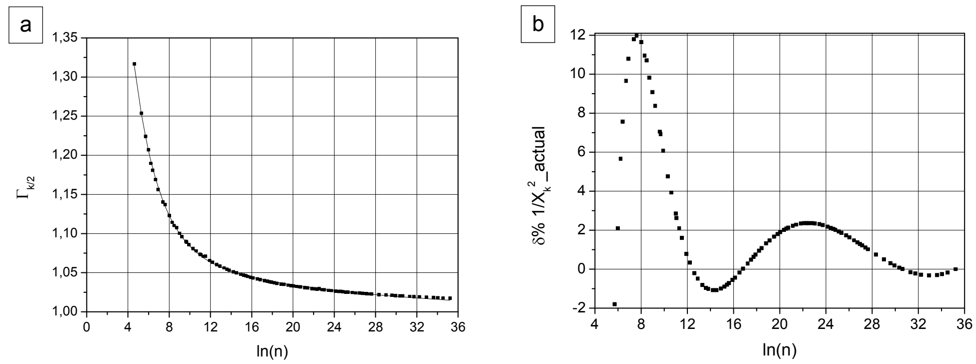


Fig. 12. a. Plot of $\boldsymbol{\Gamma_n}$vs. $\boldsymbol{ln(n)}$ b. The %$\boldsymbol{\delta}$ of the fit by $\boldsymbol{1/X_k^2(1E\!-3, n/x_o)}$

Combining the three fit functions $x_o = x_o(n)$ $k = k(n)$ and $\Gamma_{k/2} = \Gamma_{k/2}(n)$ into Eq. (1) suitable relationships can be found linking P_n/n to n. However the propagation of errors is very high in this case so that this technique deserves deeper investigations from this viewpoint too. As a matter of fact it is well known that for a set of data points fitted by an analytic function just like, for instance, the inverse of the modified chi-square function where $k = k(n)$ $\Gamma_{k/2} = \Gamma_{k/2}(n)$ $x_o = x_o(n)$ the propagation of the errors is trivially ($\Delta A = 0$ being A fix and $\Delta n = 0$ of course):

$$\Delta(1/X_k^2) = [[d(1/X_k^2)/d\Gamma_{k/2}] \cdot \Delta k + [d(1/X_k^2)/d\Gamma_{k/2}] \cdot (d\Gamma_{k/2}/dk) \cdot \Delta k + [d(1/X_k^2)/dx_o] \cdot \Delta x_o$$

so that even small errors Δk, $\Delta\Gamma_{k/2}$ and Δx_o can lead to big uncertainties on the final formula and that's why it is not shown here, apart from its complexity. Despite that, Fig. 12b shows that the resulting error might have again the features of damped oscillations around zero, that is diminishing more and more vs. ln(n): an encouraging effect.

Notwithstanding its poor precision, the technique of fitting the P_n/n values by the function $1/X_k^2(A, n/x_o)$ with the accurate choice of the two parameters k, x_o and possibly A too remains very interesting and intriguing *in primis* in that leading to the conclusion that any finite set P_n/n can be put into correspondence not only with the related $1/X_k^2(A, n/x_o)$ function but also with the associated function $C_\alpha \cdot n^\alpha$ ($\alpha > 0$) as also shown later on and that the finite sequences of prime numbers (whatever their form: $f_n = n/P_n$ $\rho_n = log(P_n)/log(n)$ P_n/n or P_n themselves) have not the property of scale invariance holding for them the scaling laws given by the modified chi-square function in one of its four forms $\pm(1/\cdot)X_k^2(A, n/x_o)$ and by the progressions $C_\alpha \cdot n^{\pm\alpha}$ and $k = 2 \pm 2\alpha$. Such results could explain the eluding and elusive nature of prime numbers.

2.5 The Fit of P_n/n by the function $C_\alpha \cdot n^\alpha$

According to the finding that any finite sequence of prime numbers can be put into correspondence with the related values of the $1/X_k^2(A, n/x_o)$ and thus with the related values of the $C_\alpha \cdot n^\alpha$ function, the last natural and obvious fit of the P_n/n values is $P_n/n \approx C_\alpha \cdot n^\alpha$ with the ad-hoc values of the parameters $C_\alpha = C(\alpha, n) = C_\alpha(n)$ and $\alpha = \alpha(n)$.

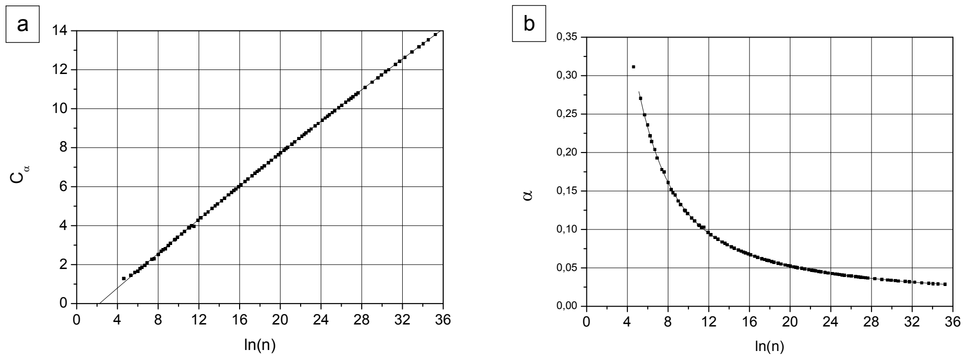


Fig. 13. a. The plot of C_α vs. $ln(n)$ b. The fit of α vs. $ln(n)$

Again the basic approach has been to choose few (~100) values of P_n/n, approximately equally distributed within the range $n \in [1E3, 2E15] \equiv [1 \cdot 10^3, 2 \cdot 10^{15}]$, thus cutting the first few primes, finding the value of $C_\alpha \cdot n^\alpha \approx P_n/n$ for any prime number, finding the values of both C_α and α and fitting all these values C_α and α by the ad hoc curve or function as reported in the Fig. 13 a for $C_\alpha = C_\alpha(n)$ and b for $\alpha = \alpha(n)$ with the fitting equations

$$C_\alpha \approx -(1.015 \pm 0.002) + (0.457 \pm 0.002) \cdot \ln(n) - (0.00105 \pm 5E-5) \cdot ln^2(n)$$

$$R^2 = 0.9999 \quad \sigma = 0.034$$

a weakly quadratic fit in ln(n) but basically a linear fit within approximately *2‰* i.e.

$$C_\alpha \approx -1.015 + 0.457 \cdot ln(n)$$

neglecting the uncertainties on the coefficients, while for $\alpha(n)$

$$\alpha \approx (0.01823 \pm 0.0014) + (0.895 \pm 0.012) \cdot n^{-1/(2.94 \pm 0.06)} + (0.154 \pm 0.008) \cdot n^{-1/(13.06 \pm 0.78)}$$
$$R^2 = 0.99983 \qquad \sigma = 6.58E-7$$

The final result is

$$P_n/n \approx C_\alpha \cdot n^\alpha \approx [-(1.015 \pm 0.002) + (0.457 \pm 0.002) \cdot ln(n) - (0.00103 \pm 5E-5) \cdot ln^2(n)] \cdot$$

$$\cdot n ** \{(0.01823 \pm 0.0014) + (0.895 \pm 0.012) \cdot n^{-1/(2.94 \pm 0.06)} + (0.154 \pm 0.008) \cdot\cdot n^{-1/(13.06 \pm 0.78)}\}$$

(where $n ** f(x) = n^{f(x)}$) leading to the % error (in comparison with the actual values of P_n/n) identical to that shown in the previous plot of Fig. 12b as it must be owing to the coincidence $1/X_k^2(A, n/x_o) = 1/[A/(2 \cdot \Gamma_{k/2}) \cdot (n/2x_o)^{k/2-1} \cdot e^{-n/2xo}] \approx C_\alpha \cdot n^\alpha$ at the utmost level as already told.

Despite a result not better than the previous ones certainly much better than the standard $n \cdot ln(n)$ and most of all a very interesting one in that showing seemingly damped oscillations around zero, once again just like the fit by $1/X_k^2(A, n/x_o$ function.

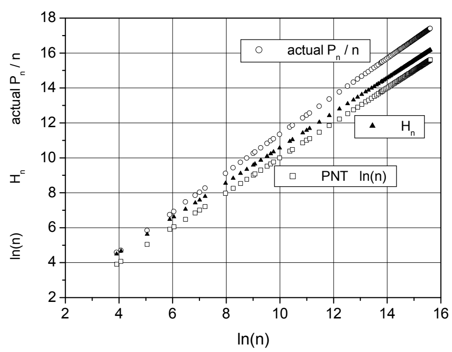
As for the future advancements, turning back to the modified chi-square function in the form $1/X_k^2(A, n/x_o$, it has to be remarked that it is a reliable and flexible function that can be used also for many further cases other than those here shown, as for instance to treat, in the same form $1/X_k^2(A, n/x_o$, the relative frequencies of the first significant digits of prime numbers and of the first two, three and four (and so on maybe) significant digits of P_n in addition to the generalized Benford's law (GBL) [61] or, in the form $X_k^2(A, n/x_o$, the sequential Bayes factors in favour of equal occurrence probabilities of the four irrational numbers $e, \pi, ln2$ and $\sqrt{2}$ [62] in order to show whether these irrational numbers are normal, that is whether or not do the 10 digits occur equally often in their decimal expansions.

2.6 The fit of P_n/n by the harmonic series

Going back to the standard prime number theorem, a refined version of it could examine the harmonic series H_n considering that the discrete function that associates the natural numbers $n \in \mathrm{N}$ with the harmonic numbers H_n is the usual logarithm function $ln(n)$ [63] so that

$$H_o = 0 \qquad H_n = \sum_{k=1 \to n} 1/k \approx ln(n) \qquad (k, n \in N \geq 1)$$

taking into account that harmonic numbers and logarithms are asymptotically convergent i.e. $\lim_{n \to \infty} H_n/\ln(n) = 1$. Thus it seems natural and trivial to examine the fit of □□/□ by the harmonic numbers i.e. $P_n/n \approx H_n = \sum_{k=1 \to n} 1/k$



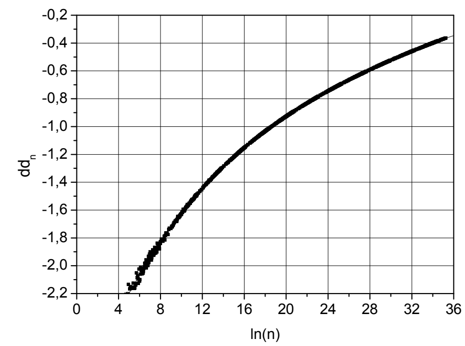


Fig. 14. a. Trends of P_n/n, H_n and $ln(n)$ b. 2nd difference $dd_n = P_n/n - (H'_n + d_n)$

Fig. 14a clarifies the actual values P_n/n compared with the discrete function H_n and the function ln(n) where it can be easily checked that the difference $d_n = H_n - P_n/n$ seems to increase so that $\lim_{n \to \infty} d_n = \infty$

However a closer look at the percent difference $\%\delta$ (not shown) reveals that starting from approximately $ln(n) \sim 13.8$ that is $n \sim 1E6$ the $\%\delta$ begins to decrease. Whatever the situation, it is easy to ascertain that the harmonic series H_n is much better than the PNT in its standard form to fit the values of P_n/n. Nonetheless, despite its reasonable results this part of the study has been dropped in that, implying the summation $H_n = \sum_{k=1 \to n} 1/k$ as the best fit function, the memory of the PC used by the author could not allow to treat values of the summation $\sum_{k=1 \to n} 1/k$ with $n > 6E6$ that is $ln(n) > 15.6$. As already told the number of $6M$ is too small to allow to draw reliable conclusions so this topic, on the other hand promising using a mainframe, has been dropped.

The same reason has compelled to drop the Euler formula for the representation of harmonic numbers expressing H_n in terms of a sum of binomials owing to the presence of factorials. As a matter of fact just at the value of $170! \sim 7.2574156153080 \ldots E + 306$ the memory of the PC used by the author fails.
It is obvious that these interesting topics could be examined only by much more powerful tools that is by a mainframe or, even better, a supercomputer.

However, a step ahead can be made using the well-known approximation for H_n with the Euler-Mascheroni constant $\gamma = 0.57721566490153\ldots$ in the form of (valid for $n \to \infty$)

$$P_n/n \approx H'_n \sim ln(n) + \gamma + 1/(2n) - 1/(12n^2) + 1/(120n^4) - 1/(252n^6) + 1/(240n^8) + O(1/n^9)$$

As usual the various stages are the same of the prior cases, that is consecutive approximations, finding the differences (and/or the percent differences) between the fit function (in this case H'_n) and the actual values of P_n/n, fitting these differences by an ad-hoc analytic function, summing up, finding the new differences, fitting them by a new ad-hoc function and so on with an iterative process up to the maximum precision attainable or up to the step in which it is no longer possible to make any fit.

The previous plot b of Fig. 14 shows the results obtained in the case of $P_n/n \approx H'_n$ for what concerns the second difference neglecting the $O(1/n^9)$ term.

The best fit of the first differences $d_n = actualP_n/n - H'_n$ vs. n (figure not shown) is:

$$d_n \approx -0.59124 + 1.98382 \cdot \left(1 - n^{-1/48.79998}\right) + 1.66071 \cdot \left(1 - n^{-1/11.01788}\right)$$

$R^2 = 0.99909 \qquad X^2_{tv} = 0.00025$

while the best fit of the second differences dd_n(Fig. 14b) is:

$$dd_n \approx -2.99128 + 0{,}18364 \cdot ln(n) - 0.00528 \cdot ln^2(n) + 5.10702E - 5 \cdot ln^3(n) + 1.0071E - 6 \cdot$$
$$\cdot ln^4(n) - 1.97307E{-}8 \cdot ln^5(n) \qquad R^2 = 0.99902 \qquad X^2_{tv} = 0.00038$$

Once again another interesting result has been got for the final stage of the second difference dd_n vs. n (plot 14b) that seems to show the asymptotic limit $\lim_{n\to\infty} dd_n = 0^-$ what allows to assume that this 2nd difference might be the end of the fit procedure with an adequate number of data so that: $P_n \approx H'_n + d_n + dd_n$

3 Future Perspectives and Developments

As a final comment it is well known that the famous relationship linking the Riemann Zeta function and the Euler product holds i.e. $\zeta(s) = \sum_{n=1\to\infty} n^{-s} = \prod_{n=1\to\infty}(1 - P_n^{-s})^{-1}$ so that prime numbers are firmly related to the non-trivial zeroes of the Zeta Riemann function and another interesting correlation linking the finite sequences of the Zeta zeroes and the finite sequences of primes is reported just as an anticipation of the next future investigations.

As a matter of fact, it has been checked that the modified chi-square function is appropriate to fit the trend of the finite sequences of the zeroes of Riemann zeta function t_n as shown in the example of next Fig. 15a and the same for the $C_\alpha n^\alpha$ function of course.

In this case (the first $100{,}000 = 100K$ t_n zeroes) the characteristics of the two fit functions $1/X^2_k(A, n/x_o$ and $C_\alpha n^\alpha$ are $A = 1E{-}06$ $k = 0.2497839450603$ $\Gamma_{k/2} = 7.54077473653752$ $x_o = 2.1657654627587E + 07$ $\alpha = 1 - k/2 = 0.87510802746985$ $C_\alpha = 3.134839902477510$

while the fit parameters between t_n and $1/X^2_k(A, n/x_o \approx C_\alpha n^\alpha$ are $R^2 = 0.99998162$

$I = 0.999934179 \qquad LSS = 6.6206E{-}03$ and $X^2_{tv} = 164.051436$

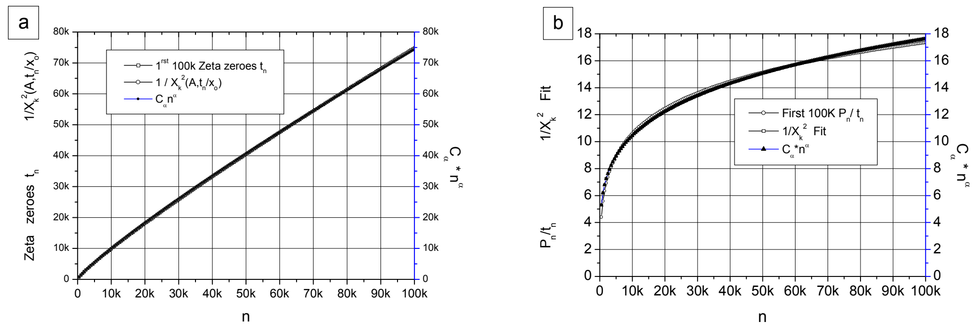


Fig. 15. a. Fits of the first 10^5 Zeta zeroes t_n b. Fits of the first $10^5 P_n/t_n$

At the present time this part of the study is still under development being rich of expectations, opening a suggestive and intriguing scenario to be furtherly deepened and expanded.

Moreover Fig. 15b shows the fit of P_n/t_n for the first $100k$ Z function zeroes up to t_{1E5} and the first *100,000* prime $P_{n=1E5}$ that is $\{P_{100K}/t_{100K}\}$ by the $1/X_k^2(A, n/x_o$ function with

$A = 1E-3$ $k = 1.545627290739160$ $\Gamma_{k/2} = 1.196211772438570$

$x_o = 1.2157612685612E + 14$

and the fit parameters

$R^2 = 0.997822$ $I = 0.995173$ $LSS = 0.461994088$ $X_{tv}^2 = 0.616423$

and by the $C_\alpha n^\alpha$ function with the values $\alpha = 0.22718635463042$ $C_\alpha = 1.2899201262216100$

and fit parameters $R^2 = 0.997822$ $I = 0.995173$ $LSS = 0.461994088$ $X_{tv}^2 = 0.655860$ despite the fact that in this case the former fit of the first $100{,}000$ Zeta zeroes t_n by the $1/X_k^2(A, t_n/x_o)$ function and by the $C_\alpha n^\alpha$ function (Fig. 15a) is much better than the latter fit of the first $100{,}000$ P_n/t_n by $1/X_k^2(A, n/x_o$ and $C_\alpha n^\alpha$ (Fig. 15b) as clearly shown by the values of the fit parameters.

4 Discussion and Concluding Remarks

Just to summarize, in conclusion the chief and leading findings of the current research are:

- the finite sets of prime numbers P_n/n (and the same holds for the P_n themselves) have not the property of scale invariance holding for them the scaling laws given by vthe modified chi-square function and the $C_\alpha n^\alpha$ function with $k = 2-2\alpha$ being

$$P_n/n \approx 1/X_k^2(A, n/x_o = 1 / \{A/(2 \cdot \Gamma_{k/2}) \cdot [n/2x_o(k)](k/2-1) \cdot e^{-n/2xo}(k)\} \approx \approx C_\alpha n \cdot^\alpha = C_\alpha \cdot n^{1-k/2}$$

- the P_n/n values are best fitted by many further kinds of analytic functions, just some of which are reported in the present study, i.e. the n^th degree polynomial in $ln(n)$, the function $\lambda(n) \cdot ln(n)$, the harmonic series $H_n = \sum_{k=1\to n} 1/k$ and its approximation H'_n as well as probably further types of functions; in addition the percent differences $\%\delta_n$or differences $d_n = actualP_n/n - ln(n)$ are fitted by the exponential decay or the (single or double) exponential growth, according to the variable examined; thus it is a must to choose, among the many possibilities, the best approximation most suitable to find the most approximate value of P_n;

- the entire methodology and all these techniques allow to get the finite value of a prime number f $P_n \approx P(n)$ starting from its counter n in many different ways though with approximations that is uncertainties;
- all the prime sequences reach their infinite limit showing well-defined patterns on their trends towards the standard asymptotical limit;
- it is implicit that a mainframe or, even better, a supercomputer could help a lot in reducing all the uncertainties first of all by examining as many prime numbers as possible;
- as it is possible to find many inductive algorithms which allow to get the approximate value of a prime number $P_n \approx P(n)$ from its counter n yet with uncertainties, so it is also conceivable to assert that prime numbers have a partial deterministic component in their behaviour without any doubt, as well as probably also a stochastic component resulting from the remainder (i.e. the difference between the $actualP_n/n$ and the fit function values i.e. the *distance* of any prime from the fit curve) yet still to be studied. This still not uncovered aspect will be the issue of next studies.

Nonetheless - though the research here shown has led to numerous interesting conclusions and results as well as to many findings all of them useful in ascertaining the nature of prime numbers - a *caveat* is necessary.

There is no doubt that there are many means and ways to describe the deterministic aspect of primes, some of which shown here, and that many of these methods may result better (or even much better) than the ones here reported. Nonetheless, what is important in the present context is the methodology implemented and the innovative process exploited i.e. fitting prime number finite sequences by analytic functions in the framework of computational/experimental mathematics and the approximation theory.

Soon after, it is the author's opinion that it has little or no sense to examine a limited number of initial prime numbers i.e. $actualP_n$ with $n < \sim 10{,}000$ as they show a seemingly random behaviour without any significant trend differently from the P_n with higher n values. The best thing to do, in order to understand the inner nature of prime numbers, is to treat as many of them as possible up to the highest achievable values of n and P_n.

Much more has to be done in the field by deepening many distinctive aspects and facets, most of all for what concerns the improvement of the calculations in order to reduce the uncertainties. Nonetheless the algorithms and the techniques here presented can open a new field of study rich of useful suggestions for number theory revealing all their power and efficacy in the future by the use of computers with faster and larger CPUs which could treat prime numbers one by one and not just one out of n as done in the present study.

Finally, as a closing remark, it is to be highlighted that at this very early stage of the investigation what is important is not just the single result attained, however remarkable it might be, but the fact that an original methodology has been introduced that may reveal itself of the utmost utility now and in the future. In other words what is noteworthy is having laid out a route towards an inner comprehension of prime numbers and their behaviour.

Competing Interests

Author has declared that no competing interests exist.

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