Inverse Monsef Distribution: Statistical Properties, Estimation and Application

Aya A.A. Mohamed [a], Mohamed M.E. Abd El-Monsef [b*], Soad M. Fergany [a] and Hamdy A. AboGabal [b]

[a] Misr University for Science and Technology, Egypt.
[b] Faculty of Science, Tanta University, Egypt.

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Original Research Article

Abstract

This paper introduces a new inverse distribution called the Inverse Monsef Distribution (IMON), which is an extension of the Monsef distribution. Most of the statistical and reliability properties of the proposed distribution are derived. The parameters are estimated using two different methods: the maximum likelihood and Cramér-von Mises methods. The proposed distribution demonstrates excellent flexibility in modeling certain types of lifetime data. To illustrate its effectiveness, three real datasets are analyzed and compared with several well-known distributions. Additionally, the reliability properties of the inverse Monsef distribution are discussed. A simulation study is conducted to evaluate the performance of the parameter estimates.

1 Introduction

The Monsef distribution (MON) and its variations, including the inverse Monsef distribution (IMON), provide flexible models for real-world data, especially in fields such as epidemiology and medical statistics. The MON distribution, introduced by Abd El-Monsef (2021) as a special case of the Erlang mixture distribution (Abd El-Monsef, 2021), has applications in modeling COVID-19 case data, breast cancer tumor sizes, and survival times for infected animals.

The weighted Monsef distribution (Alshrani & Abd El-Monsef, 2020) and the unit Monsef distribution with regression models (Abd El-Monsef et al., 2021) further broaden the utility of this family in handling diverse data types. The IMON distribution, introduced in this paper, is an extension of the MON distribution, offering improved flexibility for modeling lifetime data. Through simulation studies, the IMON distribution's parameters are effectively estimated using different methods, and applications with real data demonstrate its superior fit compared to other standard distributions.

The probability density function (pdf) for a variable distributed according to the MON distribution with scale parameter θ is defined as follows:

$$f(x,\theta) = \frac{\theta^3}{2+\theta(2+\theta)}(y+1)^2 e^{-y\theta} \qquad (1)$$

where $y \geq 0, \theta > 0$, and the cumulative distribution function is given by

$$F(x,\theta) = 1 - \frac{e^{-y\theta}}{(y+1)^2+1}\left[(\theta(y+1)+1)^2+1\right] \qquad (2)$$

If a random variable y has a Monsef distribution MON(θ) with pdf as in (1), then the random variable $x = 1/y$ is said to follow the inverse Monsef distribution with pdf

$$f(x,\theta) = \frac{\theta^3 e^{-\frac{\theta}{x}}(1+x)^2}{x^4(2+\theta(2+\theta))} \qquad (3)$$

where $x > 0, \theta > 0$, the function of the cumulative distribution (cdf) is provided by

$$F(x,\theta) = \frac{e^{-\frac{\theta}{x}}(\theta^2+2x\theta(1+\theta)+x^2(2+\theta(2+\theta)))}{x^2(2+\theta(2+\theta))} \qquad (4)$$

The proposed model gains its importance from the unimodal shape of its pdf; it can be used to model the temperature data or the rainfall amounts on wet days. Moreover, it can be used to model stock market returns, income distribution, price elasticity, traffic flow, some performance metrics in sports and many other applications in industry and medical fields.

2 Statistical Properties

In this section we will derive some statistical and reliability properties of the IMON distribution

2.1 Behavior of the density function

Theorem1 :

The pdf of the IMON distribution is unimodal for all $\theta > 0$, and it obtains its max at

$$x_0 = \frac{1}{4}\left(-4+\theta+\sqrt{16+\theta^2}\right).$$

Proof:

The first derivative of $f(x)$ is given by

$$f'(x) = \frac{-e^{-\frac{\theta}{x}}(1+x)\theta^3(2x^2 - x(-4+\theta) - \theta)}{x^6(2+2\theta+\theta^2)} \tag{5}$$

By equating at $f'(x)$ with zero, we have

$$x_0 = \frac{1}{4}\left(-4 + \theta + \sqrt{16+\theta^2}\right)$$

The second derivative of $f(x)$ given by

$$f''(x) = \frac{e^{-\frac{\theta}{x}}\theta^3\left(6x^4 - 6x^3(-4+\theta) + 2x(-5+\theta)\theta + \theta^2 + x^2(20 - 16\theta + \theta^2)\right)}{x^8(2+2\theta+\theta^2)} \tag{6}$$

By substitute the value of x_0 at the second derivative we have

$$-\frac{4096 e^{-\frac{4\theta}{-4+\theta+\sqrt{16+\theta^2}}}\theta^3\left(-32(-4+\sqrt{16+\theta^2}) + \theta(16(-4+\sqrt{16+\theta^2}) + \theta(-4(-6+\sqrt{16+\theta^2}) + \theta(-4+\theta+\sqrt{16+\theta^2})))\right)}{(2+\theta(2+\theta))(-4+\theta+\sqrt{16+\theta^2})^8} \tag{7}$$

Which is obviously less than zero. Hence, $f(x)$ is a unimodal.

The behaviors of the density function of the IMON distribution in Fig. 1, at $\theta = 3, 4, 5$.
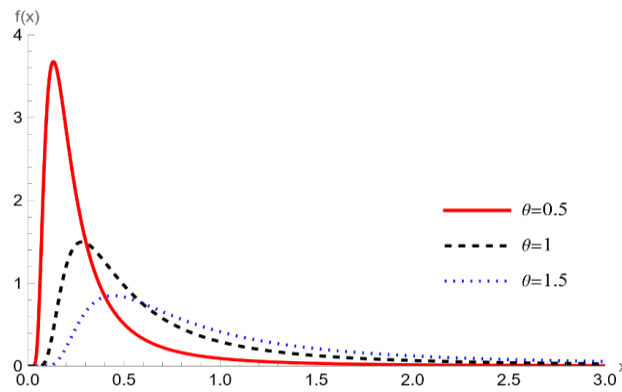

Fig. 1. Behavior of the density functions of the IMON distribution

2.2 Identifiability of the distribution

In this section, we present the identifiability of the IMON distribution. Let $\Theta_1 = \theta_1$ and $\Theta_2 = \theta_2$ be two sets of parameters $f_1(x; \Theta_1)$ and $f_2(x; \Theta_2)$ be the corresponding pdfs. From the definition of identifiability, we have,

$$f_1(x; \Theta_1) = f_2(x; \Theta_2)$$
$$\frac{\theta_1^3 e^{-\frac{\theta_1}{x}}(1+x)^2}{x^4(2+\theta_1(2+\theta_1))} = \frac{\theta_2^3 e^{-\frac{\theta_2}{x}}(1+x)^2}{x^4(2+\theta_2(2+\theta_2))} \tag{8}$$

Taking logarithms on both sides, we can write

$$\ln\theta_1^3 - \frac{\theta_1}{x} + \ln(1+x)^2 - \ln x^4 - \ln(2+\theta_1(2+\theta_1)) = \ln\theta_2^3 - \frac{\theta_2}{x} + \ln(1+x)^2 - \ln x^4 - \ln(2+\theta_2(2+\theta_2)) \tag{9}$$

$$\ln\frac{\theta_1^3}{\theta_2^3} + (\theta_2 - \theta_1)\frac{1}{x} + \ln\frac{(2+\theta_1(2+\theta_1))}{(2+\theta_2(2+\theta_2))} = 0 \tag{10}$$

This equation is equal to zero for $x > 0$ only when the coefficients are equal to zero, which is only possible when $\theta_1 = \theta_2$. Then the parameter θ is identifiable.

2.3 Reliability measures

In this section, we obtain the Reliability (survival) function, the hazard rate function. Also, the reverse hazard rate function, odds function, MillsRatio, and the mean reverse hazard (failure) rate function are obtained for the proposed distribution.

2.4 Reliability function

The Reliability function (Surles & Padgett, 2001), denoted by $R(x)$, is given by

$$R(x) = 1 - F(x) = 1 - \frac{e^{-\frac{\theta}{x}}(\theta^2 + 2x\theta(1+\theta) + x^2(2+\theta(2+\theta)))}{x^2(2+\theta(2+\theta))} \tag{11}$$

2.5 Hazard rate function

The hazard (failure) rate function $h(x)$, is also called the failure rate, instantaneous death rate for the survivors to time x during the next instant of time. The hazard rate function is given by

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{(1+x)^2\theta^3}{x^2(-\theta^2 - 2x\theta(1+\theta) + \left(-1+e^{\frac{\theta}{x}}\right)x^2(2+\theta(2+\theta)))} \tag{12}$$

Fig. 2 presents the behavior of the hazard rate function of the IMON distribution for different values of $\theta = 3, 4, 5$

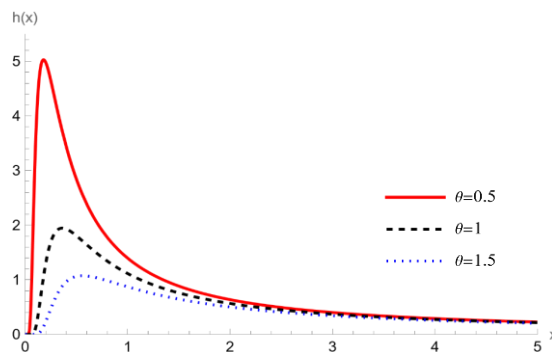**Fig. 2. Behavior of the hazard rate function of the IMON distribution**

2.6 Reverse hazard rate function

The reverse hazard (failure) rate function is defined as

$$H(x) = \frac{f(x)}{F(x)} = \frac{(1+x)^2\theta^3}{x^2(\theta^2 + 2x\theta(1+\theta) + x^2(2+\theta(2+\theta)))} \qquad x \geq 0, \ \theta > 0 \tag{13}$$

2.7 Mean Reverse hazard rate function

The mean reverse hazard (failure) rate function is

$$MR = \frac{x^2\left(\theta + x\left(2+\theta(2+\theta)\right) - e^{\frac{\theta}{x}}\theta^3\,\Gamma\left(0,\frac{\theta}{x}\right)\right)}{\theta^2 + 2\,x\,\theta(1+\theta) + x^2(2+\theta(2+\theta))} \qquad x \geq 0\,,\ \theta > 0 \qquad (14)$$

2.8 Mills Ratio (MiR)

The Mills Ratio is given by

$$\text{MiR(x)} = \frac{1}{h(x)} = \frac{x^2(-\theta^2 - 2x\theta(1+\theta) + (-1+e^{\frac{\theta}{x}})x^2(2+\theta(2+\theta)))}{(1+x)^2\theta^3} \qquad x \geq 0\,,\ \theta > 0 \qquad (15)$$

2.9 Odds function

The odds function can be written as

$$O(x) = \frac{F(x)}{R(x)} = \frac{1}{-1 + \frac{e^{\frac{\theta}{x}}x^2(2+\theta(2+\theta))}{\theta^2 + 2x\theta(1+\theta) + x^2(2+\theta(2+\theta))}} \qquad (16)$$

3 Order Statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the n ordered random sample drawn from pdf (3). Then, the density of the rth order statistic with the pdf of $X_{(r)}$ is given

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)}\sum_{k=0}^{n-r}\binom{n-r}{k}(-1)^k\,[F(x)]^{r+k-1}f(x) \qquad , x > 0 \qquad (17)$$

and the rth order cdf $F_{r:n}(x)$ is

$$F_{r:n}(x) = \sum_{j=0}^{n}\sum_{k=0}^{n-r}\binom{n}{j}\binom{n-j}{k}(-1)^k\,[F(x)]^{j+k} \qquad (18)$$

Hence, using Equations (17), (18), the pdf and the cdf of rth order statistics are, respectively, given by

$$f_{r,n}(x) = \frac{(1+x)^2\theta^3 e^{-\frac{\theta}{x}}}{x^4(2+\theta(2+\theta))B[r,1+n-r]}\left(\frac{e^{-\frac{\theta}{x}}(\theta^2 + 2x\theta(1+\theta) + x^2(2+\theta(2+\theta)))}{x^2(2+\theta(2+\theta))}\right)^{-1+r}\left(\frac{e^{-\frac{\theta}{x}}(-\theta^2 - 2x\theta(1+\theta) + (-1+e^{\frac{\theta}{x}})x^2(2+\theta(2+\theta)))}{x^2(2+\theta(2+\theta))}\right)^{n-r} \qquad (19)$$

$$F_{r,n}(x) = \sum_{j=0}^{n}\sum_{k=0}^{n-r}\binom{n}{j}\binom{n-j}{k}(-1)^k\left[\frac{e^{-\frac{\theta}{x}}(\theta^2 + 2x\theta(1+\theta) + x^2(2+\theta(2+\theta)))}{x^2(2+\theta(2+\theta))}\right]^{j+k} \qquad (20)$$

3.1 Estimation

In this section, we consider some methods of estimation and inference techniques to estimate the parameters of the proposed distribution (Bekker & Roux, 2005; Lehmann & Casella, 1998; Swain et al., 1988).

3.2 Maximum likelihood estimators (MLES)

Here, we discuss maximum likelihood estimation method for estimating the unknown parameter θ of the IMON distribution. Suppose $X_1, X_2, \ldots, X_n$ be independent and identical Then, the likelihood function based on observed sample $\{x_1, x_2, \ldots, x_n\}$, is written as

$$\prod_{i=1}^{n} f(x,\theta) = \left[\frac{\theta^3}{2+2\theta+\theta^2}\right]^n \left(e^{-\theta\sum_{i=1}^{n}\frac{1}{x_i}}\right)[\prod_{i=1}^{n}x_i{}^{-4}](\prod_{i=1}^{n}(1+x_i)^2) \qquad (21)$$

The log-likelihood function corresponding to (21) is given by

$$L(\theta) = 3n\ln(\theta) - n\ln(2 + 2\theta + \theta^2) - \theta\sum_{i=1}^{n}\frac{1}{x_i} - 4\sum_{i=1}^{n}\ln(x_i) + 2\sum_{i=1}^{n}\ln(1 + x_i) \tag{22}$$

The maximum likelihood estimator (MLES) and $\hat{\theta}$ is obtained as the simultaneous solution of the following non-linear equations:

$$\frac{\partial}{\partial\theta}L(\theta) = \frac{3n}{\theta} - \frac{n(2 + 2\theta)}{2 + 2\theta + \theta^2} - \sum_{i=1}^{n}\frac{1}{x_i} \tag{23}$$

Using (23), the MLE of θ can be obtained in terms of $\hat{\theta}$ as

$$\hat{\theta} = \frac{-A^2 + An - n^2 + 2n\bar{x}(-A - 4n + n\bar{x})}{3An\bar{x}}$$

where

$$A = (-n^3(1 + 2\bar{x}(6 + \bar{x}(24 + 5\bar{x}))) + 3\sqrt{6}\sqrt{n^6\bar{x}^2(1 + 2\bar{x}(7 + \bar{x}(27 + \bar{x}(8 + \bar{x}))))})^{1/3} \tag{24}$$

3.3 Cramer-von-mises estimators

The Cramer-von Mises estimator (CME) is $\hat{\theta}_{CME}$ of θ are obtained by minimizing

$$c(\theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_i; \theta) - \frac{2i - 1}{2n}\right)^2$$

with respect to θ, this estimator can also be obtained by solving the nonlinear equations:

$$\frac{\partial c(\theta)}{\partial\theta} = \sum_{i=1}^{n}\left(F(x_i; \theta) - \frac{2i - 1}{2n}\right)^2 F'(x; \theta) = 0$$

$$F'(x; \theta) = \frac{e^{-\frac{\theta}{x}}\theta^2(-2 - 6x(1 + x) - 2(1 + x)(1 + 2x)\theta - (1 + x)^2\theta^2)}{x^3(2 + \theta(2 + \theta))^2}$$

where $F'(x; \theta)$ is defined above.

3.4 Simulation

The performance of the various estimator techniques of the unknown parameters for the IMON (θ) is verified by simulation research (Jorda, 2010). The following is how the simulations work:

- Generate random sample with size n from the IMON distribution.
- Step 1 data are used to calculate the $\hat{\theta}$ considering the MLES, CME estimators.
- Repeat the steps 1 and 2 N times.
- Using θ and $\hat{\theta}$ to compute the Bias and the mean square errors (MSE).
- From the IMON (θ) distribution, 1000 samples were generated, where n ={20, 40, 60, 80, 100}, and by choosing θ = {0.5, 1, 1.5}. The estimators MLE, CME were evaluated in Tables 1–3
- All estimates show the property of consistency i.e. the MSE decrease as sample size increase for all parameter combinations.

where $\quad MSE_{\hat{\theta}} = E\left(\left(\hat{\theta} - \theta\right)^2\right) = Var(\hat{\theta}) + (Bias(\hat{\theta}))^2 \quad \& \quad Bias(\hat{\theta}) = E(\hat{\theta}) - \theta$

Table 1. Simulation results for $\theta = 0.5$

n		MLES	CME
20	MSE θ	0.00477	0.00691
	Bias θ	0.010253	0.01125
40	MSE θ	0.002335	0.0025
	Bias θ	0.005443	0.00624
60	MSE θ	0.001482	0.00143
	Bias θ	0.000388	0.00184
80	MSE θ	0.001102	0.00148
	Bias θ	0.001933	0.000067
100	MSE θ	0.00081	0.00061
	Bias θ	0.001175	−0.00111

Table 2. Simulation results for $\theta = 1$

N	Est	MLES	CME
20	MSE θ	0.0208415	0.04195
	Bias θ	0.022947	0.02174
40	MSE θ	0.010037	0.01033
	Bias θ	0.012119	0.01099
60	MSE θ	0.006361	0.00844
	Bias θ	0.00864789	0.00704
80	MSE θ	0.00472626	0.00510
	Bias θ	0.00436392	0.01296
100	MSE θ	0.00346287	0.00423
	Bias θ	0.0047964	-0.01544

Table 3. Simulation results for $\theta = 1.5$

n		MLES	CME
20	MSE θ	0.0520211	0.0498258
	Bias θ	0.0386468	0.0161715
40	MSE θ	0.0246393	0.023869
	Bias θ	0.0203567	0.0026093
60	MSE θ	0.0155995	0.017438
	Bias θ	0.0144589	0.019178
80	MSE θ	0.0115832	0.014064
	Bias θ	0.00743873	0.009797
100	MSE θ	0.00845126	0.011602
	Bias θ	0.00795635	-0.002503

It can be concluded that the MLE is more better than the CME as it shows minimum values of MSE and Bias for all the parameter values.

3.5 Applications

In this section, we perform the practical applicability of the proposed model using maximum likelihood estimate of the parameter to represent the potentiality of the new model as compared to some other existing life-time models by using the real data set (Anderson & Darling, 1954).

3.6 Data 1

A complete set of skewed to right data, in which 20 items are tested till failure are discussed and the values are:

List 1. Resulted value for Data 1

11.24	1.92	12.74	22.48	9.6
11.5	8.86	7.75	5.73	9.37
30.42	9.17	10.2	5.52	5.85
38.14	2.99	16.58	18.92	13.36

3.7 Data 2

The following skewed to right set of data, discussed by Nassar & Nada, (2011) with mean (13.49) greater than the median (10.6) and the skewness equals 1.61. It represents the monthly actual taxes revenue (in 1000 million Egyptian pounds) in Egypt between January 2006 and November 2010. The values are:

List 2. Resulted value for Data 2

5.9	20.4	14. 9	16.2	8.4	17.2
7.8	6.1	9.2	10.2	11	9.6
35.7	15.7	9.7	10	11.6	4.1
36	8.5	8	9.2	11.9	26.2
21.9	16.7	21.3	35.4	5.2	14.3
13.3	8.5	21.6	18.5	6.8	5.1
6.7	17	8.6	9.7	8.9	39.2
8.5	10.6	19.1	20.5	7.1	7.1
7.7	18.1	16.5	11.9	10.8	7
8.6	12.5	10.3	11.2	6.1	

Table 4. The goodness of fit measures for the data

Model	Measures								
	MLE		KS	p-value	-L	AIC	BIC	HQIC	CAIC
IMON(θ)	**9.35102**	**……**	**0.186713**	**0.488374**	**71.3471**	**144.694**	**145.69**	**144.889**	**144.916**
IGM(α,β)	6	1.4494	0.18851	.475945	71.30	146.60	148.59	146.99	147.31
IMAX(θ)	.019539	……	0.428196	.001305	80.266	162.532	163.528	162.726	162.754
IPAR(θ)	8.52173	……	0.20988	.34166	71.797	145.594	146.594	145.788	145.816
IXGM(θ)	9.16452	……	0.202687	.383896	71.6697	145.339	146.335	145.534	145.562

Table 5. The goodness of fit measures for the data

Model	Measures								
	MLE		KS	p-value	-L	AIC	BIC	HQIC	CAIC
IMON(θ)	**12.05**	**……**	**0.2892**	**0.000103**	**211.01**	**424.02**	**426.10**	**424.83**	**424.09**
IL(θ)	11.20	……	0.2906	0.00009	211.16	424.33	426.41	425.14	424.40
IGM(α,β)	.1998	13.02	0.8323	2.2×10^{-16}	314.04	632.09	636.25	633.71	632.31
IPAR(θ)	10.95	…….	0.3020	.00004	213.41	428.83	430.91	429.64	428.90
IXGM(θ)	11.79	…….	0.3003	0.00004	212.89	427.79	429.87	428.60	427.86
IWEI(α,β)	1.274	3.942	0.4995	3.2×10^{-13}	230.51	465.03	469.18	466.65	465.24

3.8 Data 3

The following data representing the rainfall amounts in Tanta reveals valuable insights into the region's climate patterns. Tanta, located in the Nile Delta of Egypt, typically experiences a Mediterranean climate, with most of its rainfall occurring during the winter months.

List 3. Resulted value for Data 3

0.529	0.266	0.506	0.260	0.250	0.069
0.136	0.130	0.531	0.265	0.254	0.107
0.232	0.195	0.264	0.129	1.219	0.221
0.125	0.119	0.447	0.124	0.154	0.207
0.227	0.682	0.172	0.635	0.152	0.165
0.044	0.147	0.128	0.209	0.085	0.263
0.192	4.013	0.131	1.459	0.361	0.186
0.344	0.119	1.758	0.197	0.461	0.065
0.110	0.574	0.130	0.123	0.190	0.142
0.080	0.186	0.090	0.690	1.086	0.342
12.710	0.412				

Table 6. The goodness of fit measures for the data

Model	Measures								
	MLE		**KS**	**p-value**	**-L**	**AIC**	**BIC**	**HQIC**	**CAIC**
IMON(θ)	**0.4628**	**……**	**0.0642**	**0.9600**	**-6.978**	**-11.957**	**-9.8301**	**-11.122**	**-11.8906**
IGM(α,β)	0.129	0.0684	0.1191	0.3426	-5.567	-7.134	-2.879	-5.4636	-6.9306
IPAR(θ)	0.5915	……	0.3145	9.37×10^{-6}	28.501	59.002	61.1298	59.837	59.069
IPMA(α, β)	0.5407	0.2300	0.1058	0.4913	-6.688	-9.377	-5.1233	-7.7072	-9.1742
IWEI(α,β)	1.3791	0.1649	0.0975	0.5964	-7.1012	-10.202	-5.9481	-8.5320	-9.9990

The IMON distribution was compared with the following distributions

- Inverse Gamma   IGM(α,β) $f(x) = \frac{\alpha e^{\frac{\beta}{x}}}{x^2} e^{(\frac{-\alpha}{\beta}(e^{\frac{\beta}{x}}-1))}$ where $x > 0$

- Inverse Maxwell   IMAX(θ)$f(x) = \frac{4}{\sqrt{\pi}}\theta^{\frac{-3}{2}}x^{-4}e^{-\theta^{-1}x^{-2}}$ where $x > 0$

- Inverse pareto   IPAR(θ) $f(x) = \frac{\theta x^{\theta-1}}{(1+x)^{\theta+1}}$ where $x > 0$

- Inverse X-Gamma   IXGM(θ) $f(x) = \frac{\theta^2}{1+\theta}\frac{1}{x^2}(1+\frac{\theta}{2}\frac{1}{x^2})e^{\frac{-\theta}{x}}$ where $x > 0$

- Inverse Lindely   IL(θ) $f(x) = \frac{\theta^2}{\theta+1}(\frac{1+x}{x^3})e^{\frac{-\theta}{x}}$ where $x > 0$

- Inverse Weibull   IWEI(α,β) $f(x) = \frac{\alpha\beta^\alpha}{x^{\alpha+1}}e^{-(\frac{\beta}{x})^\alpha}$ where $x > 0$

- Inverse power Maxwel   IPMA(α,β) $f(x) = \frac{4}{\sqrt{\pi}}\alpha\beta^{\frac{3}{2}}x^{-3\alpha-1}e^{-\beta x^{-2\alpha}}$ where $x > 0$

Tables 4, 5, and 6 indicate that the Inverse Monsef (IMON) distribution provides a better fit to the data compared to other models. The tests presented in these tables demonstrate that the IMON distribution achieves the lowest test statistics and the highest p-values when compared with seven other distributions (Al-Kzzaz & Abd El-Monsef, 2022; Eliwa et al., 2018; Jamal & Ahmad, 2022; Sharma et al., 2015; Singh & Srivastava, 2014 and Yadav et al., 2019). Therefore, the proposed distribution can be recommended as a strong alternative to the existing Monsef family of distributions.

4. Conclusion

In this paper, we propose the Inverse Monsef (IMON) distribution as an extension of the Monsef distribution, which exhibits an upside-down bathtub shape in the behavior of its density function. We derived the probability density function (PDF) and cumulative distribution function (CDF) of the IMON distribution, as well as analyzed the behavior of the PDF and hazard rate for various parameter values. Key statistical properties, including the reliability function and odds function, were also established. To estimate the unknown parameter θ\thetaθ, we applied the maximum likelihood estimation (MLE) and Cramér-von Mises methods. The flexibility and superior fit of the IMON distribution compared to other distributions were demonstrated using three datasets.

Disclaimer (Artificial Intelligence)

Authors hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

References

Abd El-Monsef, M. M. E. (2021). Erlang mixture distribution with application on COVID-19 cases in Egypt. *International Journal of Biomathematics, 14*(3), 2150015.

Abd El-Monsef, M. M., Sohsah, N. M., & Hassanein, W. A. (2021). Unit Monsef distribution with regression model. *Asian Journal of Probability and Statistics, 15*(4), 330–340.

Al-Kzzaz, H. S., & Abd El-Monsef, M. M. E. (2022). Inverse power Maxwell distribution: Statistical properties, estimation and application. *Journal of Applied Statistics, 49*(9), 2287–2306.

Alshrani, M., & Abd El-Monsef, M. M. E. (2020). The weighted Monsef distribution. *African Journal of Mathematics and Computer Science Research, 13*(2), 74–84.

Anderson, T. W., & Darling, D. A. (1954). A test of goodness of fit. *Journal of the American Statistical Association, 49*(268), 765–769.

Bekker, A., & Roux, J. (2005). Reliability characteristics of the Maxwell distribution: A Bayes estimation study. *Communications in Statistics - Theory and Methods, 34*(11), 2169–2178.

Eliwa, M. S., El-Morshedy, M., & Ibrahim, M. (2018). Inverse Gompertz distribution: Properties and different estimation methods with application to complete and censored data. *Annals of Data Science, 16*, 321–339.

Jamal, F., & Ahmad, S. (2022). Order statistics of inverse Pareto distribution. *Computational Journal of Mathematical and Statistical Sciences, 1*(1), 51–62.

Jorda, P. (2010). Computer generation of random variables with Lindley or Poisson-Lindley distribution via the Lambert W function. *Mathematics and Computers in Simulation, 81*, 851–859.

Lehmann, L. E., & Casella, G. (1998). *Theory of point estimation* (2nd ed.). Springer.

Nassar, M. M., & Nada, N. K. (2011). The beta generalized Pareto distribution. *Journal of Statistical Advances in Theory and Applications, 6*(1), 1–17.

Sharma, V. K., Singh, S. K., Singh, U., & Agiwal, V. (2015). The inverse Lindley distribution: A stress-strength reliability model with application to head and neck cancer data. *Journal of Industrial and Production Engineering, 32*(3), 162–173.

Singh, K. L., & Srivastava, R. S. (2014). Inverse Maxwell distribution as a survival model: Genesis and parameter estimation. *Research Journal of Mathematical and Statistical Sciences, 2*(7), 23–28.

Surles, J. G., & Padgett, W. J. (2001). Inference for reliability and stress-strength for a scaled Burr type distribution. *Lifetime Data Analysis, 7*, 187–200.

Swain, J., Venkatraman, S., & Wilson, J. R. (1988). Least squares estimation of distribution function in Johnson's translation system. *Journal of Statistical Computation and Simulation, 29*(4), 271–297.

Yadav, A. S., Maiti, S. S., & Saha, M. (2019). The inverse Xgamma distribution: Statistical properties and different methods of estimation. *Annals of Data Science*.