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Maxwell Equations and Magnetic Monopoles

Sebastiano Tosto

ENEA, Rome, Italy
Email: stosto@inwind.it, stosto44@gmail.com

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Abstract

The manuscript introduces an "ab initio" quantum model to deduce the Maxwell equations. After general considerations and laying out the model's theoretical framework, these equations can be derived alongside a broad variety of other results. Specifically, a corollary of the present model proposes a possible mechanism underlying the formation of magnetic monopoles and allows estimating their formation energy in order of magnitude.

Keywords

"Ab Initio" Quantum Model, Maxwell Equations, Theoretical Framework, Magnetic Monopoles, Formation Energy

1. Introduction

The Maxwell equations are the basis of classical electrodynamics; as such, however, they do not explain quantum effects such as photon-photon interaction, Planck's law and threshold feature of the photoelectric effect. The aim of this paper is to highlight how to extend the applicability range of the Maxwell equations implementing their quantum basis. After preliminary considerations in the Section 2, the Maxwell equations turn out to be in the Section 3 one of the outcomes that emphasize the generality of the proposed model. The Section 4 outlines the possible mechanism to explain the formation of magnetic monopoles and estimates an order of magnitude of formation energy. The text is organized in order to be as self-contained as possible.

2. Preliminary Conceptual Frame

This section sketches the quantum basis of the next Section 3. The initial equation is the statistical formulation of quantum uncertainty

$$\delta p_x \delta x = n\hbar = \delta\varepsilon\delta t, \tag{1}$$

introduced in [1] and obtained as a corollary of the space time constant

$$\frac{\hbar G}{c^2}=\frac{length^3}{time}. \tag{2}$$

In fact (1) are rooted on Panck length and momentum, whose product $\ell_{Pl}p_{Pl}=\hbar$ summed n times reads $n\ell_{Pl}p_{Pl}=n\hbar$. As at the left side n is factor of two Planck quantities, put $n_\ell^* n_p^*=n$: n_ℓ^* and n_p^* are arbitrary real numbers, n arbitrary integer. Thus $\delta x\delta p_x=n_\ell^* n_p^*\hbar$ putting $\delta x=n_\ell^*\ell_{Pl}$ and $\delta p_x=n_p^* p_{Pl}$. Repeating this reasoning for $\delta t\delta\varepsilon$ one finds (1), whose physical meaning is: a corpuscle of mass m having random energy ε and momentum component p_x falling within $\delta\varepsilon$ and δp_x is delocalized in δx during a time lapse δt. On the one hand (1) do not imply a specific reference sistem R: they read actually $n_\ell^* n_p^*\equiv n\equiv n_t^* n_\varepsilon^*$. On the other hand, replacing the local values of conjugate dynamical variables with the respective uncertainty ranges the physical problems are formulated overcoming both classical determinism and specific reference to any particular R. In other words, let (1) be $\delta x'\delta p'_x=n'\hbar$ in a different R': actually $\delta x\delta p_x$ and $\delta x'\delta p'_x$ are indistinguishable, because n symbolizes an arbitrary number among all allowed quantum states, *i.e.*

$$n=1,2,\cdots \quad n'=1,2,\cdots \tag{3}$$

All information of this section is rooted on (1). Rewrite now (1) as

$$\delta\varepsilon=n\hbar\omega \quad 2\pi\delta t^{-1}=\omega \Rightarrow \delta\varepsilon=\delta(h\nu), \tag{4}$$

and define $\delta\varepsilon=\varepsilon_1-\varepsilon_0$ as the energy range corresponding to all i-th quantum states $n_0\le n_i\le n_1$ of the respective $\varepsilon_0\le\varepsilon_i\le\varepsilon_1$; all ε_i are allowed by the time range δt defining ν^{-1}, while $\delta\nu=\nu_1-\nu_0=n\nu$ means that ν_1 differs from ν_0 by a discrete set of intermediate frequencies ν_i. Then (1) and (4) yield also

$$\delta\varepsilon=v_x\delta p_x \quad v_x=\frac{\delta x}{\delta t}\Rightarrow \delta\varepsilon_{kin}=\mathbf{v}\cdot\delta\mathbf{p},$$

and

$$nh\nu=nh\frac{v_x}{\lambda}=p_x v_x \quad \lambda=n\lambda_0 \quad p_x=\frac{h}{\lambda} \quad \nu=\frac{v_x}{n\lambda_0}\Rightarrow nh\nu=\mathbf{p}\cdot\mathbf{v}. \tag{5}$$

In fact $\delta x\delta p_x$ is one component of $\delta\mathbf{x}\cdot\delta\mathbf{p}$: on the one hand (1) admit in principle an arbitrary number of extra-dimensions hidden in the scalar of conjugate dynamical variables, on the other hand for any p_r allowed in δp_r also follow

$$\frac{n\hbar}{\delta r}=\frac{\hbar}{\lambda_r}\Rightarrow \begin{cases}\nearrow \ \delta p_r=\delta\left(\dfrac{\hbar}{\bar{\lambda}_r}\right)=\delta\left(\dfrac{h}{\lambda_r}\right)\\ \searrow \ 2\pi\delta r=n\lambda_r\end{cases} \tag{6}$$

The left hand side means that the size of δr is equivalent to n reduced wavelengths $\bar{\lambda}_r$, which in fact agrees at the right hand side with (6). Let us show that $\delta(h/\lambda_r)$ yields

$$h\delta\left(\lambda^{-1}\right)=\frac{\delta\left(h\nu\right)}{v_g}. \tag{7}$$

The concept of velocity is definable through (1) as $\delta x/\delta t$ or $\delta\varepsilon/\delta p_x$: the former is the usual length/time ratio, the second depends implicitly upon the number of states as actually (1) reads $\delta x/\delta t=\delta\varepsilon/\delta p_x$ skipping however $n\hbar$. Replacing the size of δx with the extent of λ, write via $v=|\mathbf{v}|$ and $p=|\mathbf{p}|$ for an e.m. wave

$$v=\frac{\lambda}{\delta t}\quad v_g=\frac{\delta\varepsilon}{\delta p}=\frac{\delta\nu}{\delta\lambda^{-1}}\Rightarrow\frac{v_g}{c}=\frac{\delta\nu}{\delta\left(\mathrm{n}\nu\right)}\quad \mathrm{n}=\frac{c}{v}\quad \nu=\frac{v}{\lambda}$$

where v_g is the group velocity of a wave packet, v the phase velocity and n the refractive index. It highlights the physical meaning of $\delta\left(1/\lambda_r\right)$ in (7).

The general concepts of energy and momentum implied in principle by (1), must be specified in fact case by case depending on the physical problem of interest; four relevant examples clarify this point.

1) Equations (4) and (6) are mere different ways to rewrite the initial (1). Implement their correlation to describe the photoelectric effect specifying first $\delta\left(eV\right)=\delta\varepsilon_{kin}$, being V the acceleration voltage acting on the charge *e*. Thus δV and $\delta\varepsilon_{kin}$ imply $\delta\left(eV\right)=\delta\varepsilon_{kin}=\delta\left(h\nu\right)$ and explain the linear trend of δV vs $\delta\nu$ with slope *h*/*e*. Also, writing $\delta\left(\varepsilon_{kin}+const\right)=\delta\left(h\nu\right)$ *i.e.* $\varepsilon_{kin}=h\nu-const$, appears also the threshold character of $const=h\nu_0$ defined by $\varepsilon_{kin}=h\left(\nu-\nu_0\right)$, *i.e.* the standard equation of the photoelectric effect with $\nu>\nu_0$. To clarify how the local values ε_{kin} and $h\nu$ are related to the respective uncertainty ranges, write explicitly

$$\frac{\varepsilon_0}{e}\le\frac{\varepsilon}{e}\le\frac{\varepsilon_1}{e}\Rightarrow V_1-V_0=\frac{\varepsilon_1}{e}-\frac{\varepsilon_0}{e}=\frac{n_1h\nu-n_0h\nu}{e}=\frac{h\nu_1-h\nu_0}{e} \tag{8}$$
$$\nu_1=n_1\nu\quad \nu_0=n_0\nu,$$

which explains in a natural way the physical meaning of stopping voltage and irradiation frequency V_0 and ν_0. Both boundaries of the ranges are arbitrary: this allows fitting any experimental irradiation conditions, e.g. intensity and frequency spectrum of the light beam, and work function $h\nu_0$ of metal irradiated. Yet (1), (7) and (6) emphasize contextually v_g and wave character of the photon too, thus overcoming Millikan's skepticism about Einstein's explanation of the photoelectric effect.

2) Note that owing to (7) and (6) and putting $v=|\mathbf{v}|$ and $p=|\mathbf{p}|$, (2) yields

$$\frac{hG}{2\pi c^2}=\frac{\lambda}{2\pi}r_m v=\frac{r_m v\delta r}{n}\quad \lambda=\frac{h}{p}\quad r_m=\frac{mG}{c^2}\quad v=\frac{p}{m}=\frac{\delta\varepsilon}{\delta p}. \tag{9}$$

These definitions are linked: *m* has been introduced to fulfill (2) via r_m and *v* involving respectively *G* and *p*. At the moment regard *m* and *v* as dimensional parameters. To examine the meaning of *m*, let us merge $p=h/\lambda\propto mv$ and call the proportionality factor β^{-1}, being β a function to be defined. Next, multiplying both sides of $h/\lambda=mv/\beta$ by *v*, one finds $h\nu/m=v^2/\beta$ being $\nu=v/\lambda$. This result yields

$$h\nu = m^* v^2 = m^* c^2 \frac{v^2}{c^2} = m^* c^2 - m^* c^2 \beta^2 \quad m^* = \frac{m}{\beta} \quad \beta^2 = 1 - \frac{v^2}{c^2} \quad p = \frac{h\nu}{v} = m^* v$$

and therefore

$$h\nu = E\frac{v^2}{c^2} \quad E = m^* c^2 = \frac{mc^2}{\beta} \quad p = m^* v = \frac{mv}{\beta} = \frac{Ev}{c^2}. \tag{10}$$

Moreover $\lambda = h/p = h\beta/mv$ yields as a limit case the length h/m^*c. More specifically, owing to (4),

$$m^* c^2 = \delta(h\nu) = nh\nu \Rightarrow \frac{c}{\nu} = \frac{nh\beta}{mc} = n\lambda_C^* \quad \lambda_C^* = \lambda_C \beta: \tag{11}$$

of course λ_C^* tends to the classical Compton length λ_C of *m* for $v \ll c$. As λ_C corresponds to $v = 0$, the physical meaning of this result is that $\lambda = c/\nu$ is the Lorentz contraction of the proper length $n\lambda_C$ for $v = 0$, which in turn implies the Lorentz time dilation too.

The fact of regarding reasonably the wavelengths as a multiple of the Compton λ_C implies that *m*, hidden in λ_C, is proportional via c^{-2} to the energy gap $h\delta\nu$. This approach merges quantum ideas, special relativity and classical physics. Putting $\beta \approx 1$ the last (9) reads in particular $p\delta p = \delta(p^2)/2$ and thus with notation analogous to (8)

$$\frac{p_1^2}{2m} - \frac{p_0^2}{2m} = \varepsilon_1 - \varepsilon_0 \Rightarrow \varepsilon_{kin} + const = \frac{p^2}{2m} \tag{12}$$

having defined constants the terms labeled with 0: the lower boundaries of the ranges play here the role of the arbitrary constant defining the energy, *i.e.* the boundary conditions. Moreover merging wave and corpuscular definitions of *p* via (2) implies owing to (6) and (9)

$$r_m = n\bar{\lambda}_m = \frac{mG}{c^2} = \frac{mG\delta t^2}{c^2\delta t^2} = \frac{mG}{\omega^2 \delta\ell^2} \quad \delta\ell = c\delta t$$
$$\Rightarrow n\omega^2 \delta r^3 = mG \quad \delta r^3 = \bar{\lambda}_m \delta\ell^2,$$

having factorized two arbitrary lengths $\bar{\lambda}_m \delta\ell^2$ into a unique δr^3. For $n = 1$ this result has the form of the classical third Kepler law. No less, the relativity is just one step further. As the constant can be in principle positive or negative, think for example to the potential energy due to an attractive constant field, one can rewrite (12) with $-const$; then multiplying side by side (12) and its modified form, one finds

$$\varepsilon_{kin}^2 - const^2 = \frac{p^4}{4m^2} \Rightarrow \varepsilon_{kin}^2 = \frac{(pc)^4}{(2mc^2)^2} + const^2; \tag{13}$$

to make this result consistent with (10), introduce an arbitrary proportionality factor ζ such that

$$pc = \zeta \frac{hc}{\lambda} = \zeta h\nu = 2\zeta mc^2 \quad mc^2 = \zeta \frac{h\nu}{2}.$$

So, not only (13) reads

$$\epsilon_{kin}^2 = \left(p'c\right)^2 + const^2 \quad p' = \zeta p, \tag{14}$$

analogous to and consistent with that inferred from (10), but also the rest energy mc^2 appears to be equivalent to the zero point delocalization energy of a harmonic oscillator with quantized frequencies $\zeta\nu$.

Note now that with $n\lambda$ quantized in (6), (11) yields

$$h\nu = \frac{mc^2}{n} = m_0 c^2 \quad m_0 = \frac{m}{n} \tag{15}$$

whose meaning is not merely formal: (9) suggests introducing the arbitrary mass m_0 to write

$$\frac{\hbar G}{c^2} = r_{m_0} \nu \delta r \quad r_{m_0} = \frac{r_m}{n}. \tag{16}$$

The third Kepler law takes the usual form as a function of m_0 of (15), which also implies that r_m expressed as a function of m_0 reads in turn, quoting for brevity two numbers n of states only,

$$r_m = \frac{n m_0 G}{c^2} = \begin{cases} \nearrow \dfrac{m_0 G}{c^2} & n = 1 \\ \searrow \dfrac{2 m_0 G}{c^2} & n = 2 \end{cases} \tag{17}$$

The fact that in (16) δr is quantized shows that the meaning of (6) goes well beyond its early purpose of regarding the electron as a wave interacting with a nucleus; thus $\lambda\!\!\!^{-}$ plays the role of reference space length whose extent is analogous to the range size δx in (1). Once more corpuscular and wave interpretations are compatible and equivalent.

These definitions, inferred via the physical dimensions of (2) through the quantization of λ only, are further concerned in the Section 5. Note here that (9) merge wave and corpuscular character of momentum, and provide further information. As $p\delta p = \delta\left(p^2\right)/2 = -\left(h^2/\lambda^3\right)\delta\lambda$, write then

$$\frac{\delta\lambda}{\lambda} = -\frac{1}{2}\frac{\delta\left(pc\right)^2}{\left(h\nu\right)^2} \Rightarrow \nu\delta\frac{1}{\nu} = -\frac{1}{2}\frac{\delta\left(pc\right)^2}{\left(h\nu\right)^2} \tag{18}$$

that yields

$$\frac{\delta\varphi}{c^2} = -\frac{1}{2}\frac{\delta\left(pc\right)^2}{\left(h\nu\right)^2} \Rightarrow \varphi = -\frac{1}{2}\frac{\left(pc\right)^2}{\left(h\nu/c\right)^2} = -\frac{\left(pc\right)^2}{2p'^2} \quad p' = \frac{h\nu}{c}.$$

Owing to (18) is relevant the result, with the notation (8),

$$\frac{\varphi_1 - \varphi_0}{c^2} = -\frac{\nu_1 - \nu_0}{\nu_0} \quad \nu = \nu_1. \tag{19}$$

Physical dimensions and sign of φ suggest its meaning of potential energy per unit mass; $\delta\varphi$ is related to $\delta\nu$ with respect to arbitrary reference values of ν_0 and corresponding p'. The classical approximation of φ is easily guessed replacing $p \approx mv$, which yields $\varphi_{cl} \approx -v^2/2$. As owing to (9)

$2\varphi_{cl}/c^2 = -MG/rc^2 = -r_M/r$ is the potential energy at of a body at a distance r from M. The result (19) is sensible: e.g. elementary considerations show that the classical escape velocity of a body r apart from M is $\approx \sqrt{2MG/r}$.

Consider eventually the following chains of equations; owing to (12), (16) and (17)

$$h\frac{\hbar G}{c^2} = \epsilon\mathcal{V} = \frac{p^2}{2m}\mathcal{V} = \frac{h^2}{2m\lambda^2}\mathcal{V} \quad h\mathcal{V} = \frac{2m\lambda^2\hbar G}{c^2}$$
$$\Rightarrow r_{bh} = \frac{2mG}{c^2} \quad \mathcal{V} = \lambda\!\!\!^{-2} 2\pi r_{bh} = \lambda\!\!\!^{-2} n_{bh}\lambda_{bh}, \tag{20}$$

being ϵ and $\mathcal{V}$ arbitrary energy and volume by dimensional reasons. Moreover the first (9) reads identically

$$\frac{\hbar G}{c^2} = \lambda\!\!\!^{-} r'_m v' \quad r'_m = \frac{r_m}{\beta'} \quad v' = v\beta' \Rightarrow p = mv = \frac{mv'}{\beta'} \quad m = \frac{m'}{\beta'}, \tag{21}$$

being β' a function to be defined. The shortest way to examine (21) is to put $\beta' = \sqrt{1 - v'^2/c^2}$. Then

$$v^2 = \frac{v'^2}{1 - v'^2/c^2} \Rightarrow v'^2 = \frac{v^2}{1 + v^2/c^2} \quad p' = \frac{m'v'}{\sqrt{1 - v'^2/c^2}} \quad \epsilon' = \frac{m'c^2}{\sqrt{1 - v'^2/c^2}};$$

replacing next $v' = \sqrt{v^2 v_4/(v_2 + v_3)}$ and $v = \sqrt{v_2 v_3}$, all velocities are arbitrary, one finds via trivial steps

$$v_4 = \frac{v_2 + v_3}{1 + v_2 v_3/c^2} \Rightarrow v_4 \le c. \tag{22}$$

These preliminary results are not surprising. They emphasize how mass, time and length already inherent the physical dimensions of the constants defining (2) are extracted explicitly or recombined implicitly via the positions (9), (10) and (20).

3) Eventually follows in this frame based on (1) and (2) only also the quantization of the electric charge. Let q be an arbitrary amount of charge and q_0 a reference charge, e.g. that of the electron. Write then by dimensional reasons $\varepsilon = q^2/r$ in the c.g.s. system; so, differentiating ε one finds owing to (1)

$$\delta\varepsilon\delta t = -\frac{q^2}{r^2}\delta r\delta t = n\hbar = \frac{q}{q_0}\hbar \quad n = \frac{q}{q_0}. \tag{23}$$

The last position is now checked. Depending on the sign of q_0, it follows

$$\pm\frac{q_0 q}{r}\frac{\delta r}{r} = \frac{\hbar}{\delta t} \Rightarrow \pm\frac{q_0^2}{r}\frac{\delta r}{r} = \frac{\hbar}{n\delta t}$$
$$\Rightarrow \pm\frac{q_0^2}{r^2} = \frac{\delta p_r}{n_r n\delta t} = \frac{F_r}{n_r n} \quad F_r = \frac{\delta p_r}{\delta t} \quad \delta p_r = \frac{n_r\hbar}{\delta r} \tag{24}$$

i.e. the radial Coulomb force F_r via charge quantization (23).

4) This subsection concerns more specifically the way to find the Maxwell equations.

As *momentum*/*volume* = *flux* = *mass*/(*surface* × *time*) by dimensional reasons, consider an arbitrary surface $A = A(y,z)$ normal to the x-axis crossed by a flux of particles of total mass m, initially assumed moving at the same velocity v_x; thus $p_x v_x$ of (7) introduces $p_x v_x/\mathcal{V}$ that in turn defines the flux component j_x per unit surface and time of the particles crossing $A(y,z)$ delocalized in $\mathcal{V}$ with momentum p_x. Thus

$$j_x = \frac{p_x}{\mathcal{V}} \quad \varepsilon = p_x v_x \Rightarrow \varepsilon = \mathbf{p}\cdot\mathbf{v} \quad \mathbf{j} = \frac{mv}{\mathcal{V}} \tag{25}$$

yield

$$\frac{p_x}{\mathcal{V}} = j_x = v_x\rho \quad \rho = \frac{m}{V}; \tag{26}$$

the component notation of $\mathbf{v}$ and $\mathbf{p}$ is used to remark their reference to (1). Note that $p_x/\mathcal{V} = v_x\rho$ implies

$$\frac{p_x}{\mathcal{V}} = \frac{mv_x}{\mathcal{V}} = \frac{mv_x c^2}{c^2\mathcal{V}} \Rightarrow p_x = \frac{v_x\epsilon}{c^2} \quad \epsilon = mc^2, \tag{27}$$

which clarifies that in fact $p_x = mv_x$ holds provided that m is replaced by ϵ/c^2 in agreement with (9).

The results (27) and (4) along with (22) and (10) exemplify the chance of obtaining via (1) quantum and relativistic outcomes even starting from a classical conceptual frame. Also, (26) relates δp_x of (1) to the corresponding $\delta j_x = v_x\delta\rho + \rho\delta v_x$; differentiating (26) one finds

$$\delta\left(\frac{p_x}{\mathcal{V}}\right) = \delta(\rho v_x) = \delta j_x \quad \frac{\delta j_x}{\delta x} = v_x\frac{\delta\rho}{\delta x} + \rho\frac{\delta v_x}{\delta x}. \tag{28}$$

Therefore the left hand side of (28) reads, owing to (26),

$$\begin{gathered}\delta j_x = \frac{p_{x2}}{\mathcal{V}_2} - \frac{p_{x1}}{\mathcal{V}_1} = j_{x2} - j_{x1} = \rho_2 v_{x2} - \rho_1 v_{x1} \\ \Rightarrow \mathbf{j}_2 - \mathbf{j}_1 = \rho_2\mathbf{v}_2 - \rho_1\mathbf{v}_1 \quad \delta\mathbf{j} = \delta(\rho v),\end{gathered} \tag{29}$$

which in turn admit two chances formally identical:

$$\begin{gathered}\frac{p_{x2}}{\mathcal{V}_2} = j_{x2} = \rho_2 v_{x2} \quad \frac{p_{x1}}{\mathcal{V}_1} = j_{x1} = \rho_1 v_{x1} \\ \text{or} \quad \frac{p_{x2}}{\mathcal{V}_2} = -j_{x1} = -\rho_1 v_{x1} \quad \frac{p_{x1}}{\mathcal{V}_1} = -j_{x2} = -\rho_2 v_{x2}\end{gathered} \tag{30}$$

however different from a physical point of view. This appears rewriting (28) according to $j_x = \pm\rho v_x$ as

$$\frac{\delta(p_x/\mathcal{V})}{\delta x} \Rightarrow \pm\frac{\delta j_x}{\delta x} = \pm v_x\frac{\delta\rho}{\delta x} \pm \rho\frac{\delta v_x}{\delta x}: \tag{31}$$

reasonably the x-components v_x of $\mathbf{v}$ and its change δv_x can take in principle both signs. Merging (29) and (31), $\mathbf{j}$ and $\nabla\cdot\mathbf{j}$ read in general

$$\mathbf{j} = \rho v \quad \nabla\cdot\mathbf{j} = \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v} = \pm\dot{\rho} \pm \rho\nabla\cdot\mathbf{v} \quad \dot{\rho} = \sum_i \frac{\partial\rho}{\partial x_i}\frac{\partial x_i}{\partial t} = \mathbf{v}\cdot\nabla\rho \tag{32}$$

and thus, according to (26), also

$$\mathbf{j} = \frac{p}{\mathcal{V}} \quad \mathbf{r}\times\mathbf{j} = \frac{\mathbf{r}\times\mathbf{p}}{\mathcal{V}} = \frac{\mathbf{M}}{\mathcal{V}} \quad \delta\mathbf{j} = \frac{\delta\mathbf{p}}{\mathcal{V}} - \frac{\mathbf{j}}{\mathcal{V}}\delta\mathcal{V}. \tag{33}$$

It is known that in fluid dynamics the divergence of velocity vector implies the rate of time change of a moving fluid element per unit volume: so $\nabla\cdot\mathbf{v}=0$ concerns a stationary model where $\mathcal{V}=const$ and the fluid is incompressible [2]. Instead $\nabla\cdot\mathbf{v}\neq 0$ implies non-conservation of mass moving out of a differential volume $d\mathcal{V}$ during dt, which means decreasing amount of mass in $\mathcal{V}$ with positive divergence of stream density. The notation $\mathbf{j}_1$ and $\mathbf{j}_2$ has been so far implemented, e.g. in (29), to introduce $\delta\mathbf{j}$ and justify $\nabla\cdot\mathbf{j}$ with the formalism of the uncertainty ranges starting from (1). Now to implement (33) is more significant the notation $\mathbf{j}_+$ and $\mathbf{j}_-$ that reminds the signs of (32); considering separately both chances allowed for $\nabla\cdot\mathbf{j}$, these equations with the minus sign agree with Fick's law

$$\nabla\cdot\mathbf{j}_-+\dot{\rho}=0\quad\nabla\cdot\mathbf{v}_-=0. \tag{34}$$

Moreover this reasonable result pairs with the further chance

$$\nabla\cdot\mathbf{j}_++\dot{\rho}=\sigma\quad\sigma=\rho\nabla\cdot\mathbf{v}_++2\dot{\rho}\neq 0. \tag{35}$$

Realistically $\mathbf{v}_-\neq\mathbf{v}_+$, as their physical meaning is different too: $\mathbf{v}_-$ fulfills the continuity Equation (34), $\mathbf{v}_+$ in general does not, except in a specific case where $\mathbf{v}_+^*$ verifies in particular

$$\nabla\cdot\mathbf{v}_+^*+2\dot{\rho}/\rho=0. \tag{36}$$

Consider that (1) require introducing range boundaries, regardless of the random local values x and p_x included in the respective ranges; this holds in particular for the dynamical variables in (26). Therefore, once considering only the boundary values of j_x, as in (1), becomes inessential the condition of equal v_x for all particles delocalized in $\mathcal{V}$ concurring to the whole value of m: indeed appear in (29) the velocity boundary values $\mathbf{v}_1$ and $\mathbf{v}_2$, regardless of the actual velocity distribution of the i-th components $v_{x_i1}\leq v_{xi}\leq v_{x_i2}$. Now define $\mathcal{V}$ via an arbitrary surface A such that

$$\rho A=\frac{m}{\delta x}\quad A=A(y,z)\quad\delta x=x-x_0 \tag{37}$$

and calculate $\partial(\rho A)/\partial x$; in agreement with (27) one finds

$$-A\frac{\partial\rho}{\partial x}=\frac{m}{\delta x^2}=\frac{m}{\delta x^2\delta t}\delta t\;\Rightarrow\;-\frac{A}{\delta t}\frac{\partial\rho}{\partial x}=\frac{\delta x}{\delta t}\frac{m}{\delta x^3}=\frac{mv_x}{\delta x^3},$$

which reads, owing to (25),

$$-D\frac{\partial\rho}{\partial x}=v_x\rho=j_x\quad D=\frac{A}{\delta t}\quad\rho=\frac{m}{A\delta x}\;\Rightarrow\;\mathbf{j}=-D\nabla\rho. \tag{38}$$

Clearly D is the diffusion coefficient governing the flow of m through the surface A due to the mass density gradient $\nabla\rho$; the sign of (38) agrees with the first Fick law. The connection of (38) with D is not surprising owing to (36), which in fact yields

$$\frac{\rho}{\rho_0}=\exp\left(-\frac{t\nabla\cdot\mathbf{v}_+^*}{2}\right)\;\Rightarrow\;\frac{\rho\kappa}{\rho_0\kappa}=\frac{D}{D_0}\exp\left(-\frac{t\nabla\cdot\mathbf{v}_+^*}{2}\right)=\frac{D}{D_0}\exp\left(-\frac{\epsilon_{act}}{\epsilon}\right) \tag{39}$$

being ρ_0 integration constant and κ an appropriate dimensional constant. Define indeed

$$\mathbf{v}_+^* = \frac{2}{3}\frac{\mathbf{r}\tau}{\tau'(\tau''+t)}$$

so that $\mathbf{v}_+^*$ does not diverge for $t \to 0$ or $t \to \infty$. Replacing in (39) one finds

$$\frac{D}{D_0} = \exp\left(-\frac{t}{t+\tau''}\frac{\epsilon'}{\epsilon}\right) \quad \epsilon' = \frac{2}{3}\frac{h}{\tau'} \quad \epsilon = \frac{h}{\tau}: \tag{40}$$

the exponential tends to the constant ratio of two energies that with appropriate values of the arbitrary constants τ and τ' is in principle compatible with the aforesaid ϵ_{act}/kT. After an initial transient, arbitrarily short depending on the time lapse τ'', (40) takes the usual form of activation energy ϵ_{act} driven dependence of D upon the temperature T.

This result follows in the particular case where holds (36) despite, in general, $\mathbf{v}_+$ fulfills (35). It appears also noting that $\rho\nabla\cdot\mathbf{v}_- \equiv 0$ anyway, whereas for $\mathbf{v}_+$ it applies for $\rho\nabla\cdot\mathbf{v}_+^* = -2\dot{\rho}$ only; *i.e.*

$$\nabla\cdot(\rho\nabla\cdot\mathbf{v}_-) \equiv 0 \quad \nabla\cdot(\rho\nabla\cdot\mathbf{v}_+) \neq 0. \tag{41}$$

- On the one hand, it is interesting to note that

$$\frac{\partial}{\partial t}\frac{\partial j_x}{\partial v_x} = -\frac{\partial}{\partial t}\frac{\partial(\rho v_x)}{\partial v_x} = -\dot{\rho} \quad \frac{\partial j_x}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{D\partial\rho}{\partial x}\right) \Rightarrow \nabla\cdot\mathbf{j} = -\nabla\cdot(D\nabla\rho); \tag{42}$$

therefore

$$\dot{\rho} = \nabla\cdot(D\nabla\rho) \Leftrightarrow \frac{\partial}{\partial t}\frac{\partial j_x}{\partial v_x} = \frac{\partial j_x}{\partial x} \tag{43}$$

i.e. j_x fulfills the Lagrangian because the left hand side is the second Fourier equation that concerns the heat diffusion equation.

- On the other hand, if (38) and (32) are correct, then even the first equality (38) should have its own identifiable physical meaning. To check this point note that

$$\frac{d\rho}{\rho} = -\frac{v_x dx}{D} \Rightarrow \frac{\rho}{\rho'} = \exp\left(-\frac{v_x\delta x}{D}\right), \tag{44}$$

having assumed for simplicity v_x constant. Of course nothing hinders regarding $\rho' = \rho_0 \pm \rho$, so that

$$\frac{\rho'}{\rho} = \exp\left(\frac{v_x\delta x}{D}\right) \Rightarrow \frac{\rho_0}{\rho} \pm 1 = \exp\left(\frac{v_x\delta x}{D}\right) \quad \rho' = \rho_0 \pm \rho$$

yields

$$\rho = \frac{\rho_0}{\exp\left(\frac{v_x\delta x}{D}\right) \mp 1} = \frac{\rho_0}{\exp\left(\frac{mv_x\delta x/\tau}{mD/\tau}\right) \mp 1} \Rightarrow \rho = \frac{\rho_0}{\exp\left(\frac{\delta\epsilon_n}{\epsilon_0}\right) \mp 1}: \tag{45}$$

the physical meaning of (45), merely inferred with the help of dimensional reasoning, is recognizable considering that any result obtained from (1) actually refers to the n-th quantum state defined by uncertainty ranges of dynamical variables: it explains the notation $\delta\varepsilon_n$ related to $(m/\tau)v_x\delta x$. The analytic form of (45) corresponds to two possible statistical distributions of energy of particles in the n-th quantum state with respect to the reference energy ϵ_0 uniquely defined by mD/τ for an arbitrary m.

An analogous reasoning concerns the differential $d\rho$ of (44). Consider now

$$d\rho = \rho\frac{d\rho}{\rho} = \rho d\log\rho = \rho\left(\log(\rho + d\rho) - \log\rho_0\right) = \rho\log\frac{\rho'}{\rho_0} \quad \rho' = \rho + d\rho.$$

Regard now the generic ρ as the value pertinent to the n-th allowed quantum states; then

$$\rho \to \rho_n \quad d\rho \to d\rho_n \Rightarrow d\rho_n = \rho_n\log\frac{\rho'_n}{\rho'_0} = \rho'_0\frac{\rho_n}{\rho'_0}\log\frac{\rho'_n}{\rho'_0} \quad \rho'_0 = const$$

which means

$$\frac{d\rho_n}{\rho'_0} = w_n\log w_n \quad w_n = \frac{\rho'_n}{\rho'_0} \Rightarrow S = -\sum_n\frac{d\rho_n}{\rho'_0} = -\sum_n w_n\log w_n. \tag{46}$$

In addition to (45) one infers the statistical formulation of the classical entropy.

Equations (42), (45) and (46) have been explicitly introduced to emphasize that n is not mere quantum number, but the number of quantum states allowed to any physical system. This feature of n implies a additional relevant corollary concerned in the Section 5. Anyway, the basic considerations and ancillary results hitherto exposed assure the generality and validity of this theoretical framework.

At this point regard (1) as a reliable quantum basis to introduce the specific theoretical frame bringing to the Maxwell equations. It is clear that (38) and (32) hold also for a distribution of n_e electric charges simply replacing m with $n_e e$: multiplying by $n_e e/m$ both sides of the last (38), the mass flow $\mathbf{j}$ turns into charge flow $\mathbf{J}$ of n_e charges in $\mathcal{V}$ displacing at average rate $\mathbf{v}$ while ρ is from now on charge density in $\mathcal{V}$.

3. The Maxwell Equations

Consider two vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ corresponding to and inferred from $\mathbf{J}_1$ and $\mathbf{J}_2$. As $\mathbf{U}_1$ and $\mathbf{U}_2$ must be compliant with (32), (33) and (34), plug reasonably both vectors in the same conceptual frame proven consistent with (27), (42), (43), (45) and (46).

Define first the correspondence of $\mathbf{J}_1$ with $\mathbf{U}_1$ via (34) putting $\mathbf{J}_1 = \nabla\times c\mathbf{U}_1$, which yields

$$\nabla\cdot(\nabla\times c\mathbf{U}_1) = 0 = \nabla\cdot\mathbf{J} + \dot{\rho} \quad \mathbf{J}\equiv\mathbf{J}_-; \tag{47}$$

so $\mathbf{J}$ fullfills (34). Also, guess the correspondence of $\mathbf{J}_2$ with $\mathbf{U}_2$ defining

$$\nabla\cdot\mathbf{U}_2 = \rho \tag{48}$$

in order that

$$\nabla \cdot \frac{\partial \mathbf{U}_2}{\partial t} = \dot{\rho}; \tag{49}$$

once having expressed $\mathbf{J}_1$ and $\mathbf{J}_2$ via $\mathbf{U}_1$ and $\mathbf{U}_2$, (47) reads owing to (49)

$$\nabla \cdot \left(\nabla \times c\mathbf{U}_1 \right) = \nabla \cdot \mathbf{J} + \nabla \cdot \frac{\partial \mathbf{U}_2}{\partial t}. \tag{50}$$

Next, eliminate $\nabla \cdot$ from this equation, in order that (50) is not trivial equality of terms identically null. Rewrite indeed $\nabla \cdot \left(\nabla \times c\mathbf{U}_1 - \mathbf{J} - \partial \mathbf{U}_2 / \partial t \right) = 0$ as

$$\nabla \times \left(\mathcal{T} + c\mathbf{U}_1 \right) - \mathbf{J} - \frac{\partial \left(\mathcal{X} + \mathbf{U}_2 \right)}{\partial t} = \mathbf{Q}: \tag{51}$$

the arbitrary vector fields $\mathbf{Q}, \mathcal{T}, \mathcal{X}$, dutifully introduced for sake of generality, are definable as

$$\mathbf{Q} = \mathbf{Q}(t) \quad \mathcal{T} = \mathcal{T}(t) \quad \mathcal{X} = \mathcal{X}(x, y, z) \tag{52}$$

but also less restrictively as

$$\mathbf{Q} = \nabla \times \mathbf{Q}'(x, y, z, t) \quad \mathcal{T} = \mathcal{T}'(x, y, z, t) \quad \nabla \times \mathcal{T}'(x, y, z, t) = \frac{\partial \mathcal{X}'(x, y, z, t)}{\partial t}. \tag{53}$$

The vector fields $\mathcal{T}, \mathcal{X}, \mathbf{Q}$ and $\mathcal{T}', \mathcal{X}', \mathbf{Q}'$ are mere consequence of the positions (47) and (48), neither of which requires "ad hoc" hypotheses additional to the charge conservation (34). Despite the mathematical implications of (51) and (53) would deserve a separate discussion, e.g. to infer the Lorentz condition, attention is focused now on the more essential (50) putting for brevity

$$\nabla \times c\mathbf{U}_1 = \mathbf{J} + \frac{\partial \mathbf{U}_2}{\partial t} \quad \mathcal{T}, \mathcal{X}, \mathbf{Q} = 0 \quad \mathcal{T}', \mathcal{X}', \mathbf{Q}' = 0. \tag{54}$$

Regard $\mathbf{U}_1$ as a sum of two fields and $\mathbf{U}_2$ as a difference of two fields, say preliminarily for a more immediate and simple assessment of (51)

$$\mathbf{U}_2 = \mathbf{E} - \mathbf{H} \quad \mathbf{U}_1 = \mathbf{E} + \mathbf{H}; \tag{55}$$

at both right hand sides appear two combinations of the same $\mathbf{E}$ and $\mathbf{H}$ fields for simplicity, being clearly unnecessary and redundant to introduce further fields additional to that of (51) and (53). So, implementing (48) and the simplified form (54) of (50), one finds

$$\nabla \cdot \left(\mathbf{E} - \mathbf{H} \right) = \rho \quad \nabla \times c\left(\mathbf{E} + \mathbf{H} \right) = J + \frac{\partial \left(\mathbf{E} - \mathbf{H} \right)}{\partial t}: \tag{56}$$

the first (56) yields

$$\nabla \cdot \left(\mathbf{E} - \nabla \times \mathbf{A} \right) = \nabla \cdot \mathbf{E} = \rho \quad \mathbf{H} = \nabla \times \mathbf{A} \quad \nabla \cdot \mathbf{H} = 0, \tag{57}$$

whereas the second (56) splits in turn as

$$\nabla \times c\mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t} \quad \nabla \times c\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}. \tag{58}$$

It appears that (58) plus the two ones deductible from the first (56), *i.e.* $\nabla \cdot c\mathbf{E} = \rho$ and $\nabla \cdot c\mathbf{H} = 0$, are closely related to the Maxwell equations, which are therefore inferred from (1) through the steps (26) to (34). In summary the equations of interest are (57) and (58), which read in the c.g.s. system

$$\nabla\cdot\mathbf{E}=4\pi\rho_e \quad \nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \quad \nabla\times\mathbf{H}=\frac{1}{c}\left(\frac{\partial\mathbf{E}}{\partial t}+4\pi\mathbf{J}\right) \quad \nabla\cdot\mathbf{H}=0; \tag{59}$$

the factor 4π in the first equation, which results from $\rho=4\pi\rho_e$ and therefore appears also in the definition of $\mathbf{J}$, is due to the Gauss theorem fulfilled by ρ_e in agreement with (34).

In this respect two further chances in principle possible to split the second (56) are dutifully worth noting:

$$\nabla\times c\mathbf{E}=\frac{\partial\mathbf{E}}{\partial t} \quad \nabla\times c\mathbf{H}=-\frac{\partial\mathbf{H}}{\partial t}+\mathbf{J} \tag{60}$$

or

$$\nabla\times c\mathbf{E}=\frac{\partial\mathbf{E}}{\partial t}+\mathbf{J} \quad \nabla\times c\mathbf{H}=-\frac{\partial\mathbf{H}}{\partial t}. \tag{61}$$

These equations have their own physical meaning alternative to (58) and (57); in principle there is no reason to exclude these chances, which however have scarce physical interest. Indeed (60) and (61) actually concern separate fields, either $\mathbf{E}=\mathbf{E}(x,y,z,t)$ and $\mathbf{H}=\mathbf{H}(x,y,z,t,\mathbf{J})$ or $\mathbf{H}=\mathbf{H}(x,y,z,t)$ and $\mathbf{E}=\mathbf{E}(x,y,z,t,\mathbf{J})$. The solutions of these equations, whatever they might be, would provide space and time profiles of independent magnetic and electric fields: instead, only combining these fields as in (56) and (58), even with the mere (54) one actually introduces via (55) the e.m. field and finds in fact the classical Maxwell equations (59).

Some further considerations on this approach deserve attention.

1) Are significant the definitions of $\mathbf{J}$ in the Section 2, in particular the double signs in (32).

- In this regard the first and third (59) yield, according to (32) expressed as a function of charges,

$$\nabla\cdot\frac{\partial\mathbf{E}}{\partial t}=4\pi\dot{\rho} \Rightarrow \nabla\cdot(\nabla\times\mathbf{H})=0=\frac{1}{c}(4\pi\dot{\rho}+4\pi\nabla\cdot\mathbf{J})=\frac{4\pi}{c}(\dot{\rho}\pm\dot{\rho}\pm\rho\nabla\cdot\mathbf{v}): \tag{62}$$

i.e. the right hand side of (62) reads, in agreement with (35) and (34),

$$\text{with plus sign}\rightarrow\rho\nabla\cdot\mathbf{v}_+=-2\dot{\rho} \quad \text{or} \quad \text{with minus sign}\rightarrow\rho\nabla\cdot\mathbf{v}_-=0. \tag{63}$$

- The minus sign in (62) implies the continuity equation of electric charges *i.e.* a stationary model where the volume element $\mathcal{V}=const$ enclosing the charges is incompressible; owing to (34), (62) yields

$$\nabla\cdot(\nabla\times\mathbf{H})=0=-\frac{4\pi}{c}\rho\nabla\cdot\mathbf{v}_-. \tag{64}$$

- The plus sign allows obtaining an analogous form of (62) assuming $2\dot{\rho}+2\rho\nabla\cdot\mathbf{v}_+=0$: this does not contradict (35), which requires $\sigma=2\dot{\rho}+\rho\nabla\cdot\mathbf{v}_+=-\rho\nabla\cdot\mathbf{v}_+\neq 0$. So the first (63) yields

$$\nabla\cdot(\nabla\times\mathbf{H})=0=\frac{4\pi}{c}(2\dot{\rho}+2\rho\nabla\cdot\mathbf{v}_+) \quad 2\dot{\rho}+2\rho\nabla\cdot\mathbf{v}_+=0. \tag{65}$$

The left hand side is fulfilled by $\nabla\times\mathbf{H}=0$ itself, but in principle even by an arbitrary $\nabla\times(\mathbf{H}+\mathbf{H}')$. A possible way to rewrite the first (65) is then

$$\nabla\cdot\nabla\times\mathbf{H}'=\frac{4\pi}{c}\left(2\dot{\rho}+2\rho\nabla\cdot\mathbf{v}_+\right)=0; \tag{66}$$

it fullfils (65), (66) and $\mathbf{H}'$ itself. The second (63) is a particular case of the first one for $\dot{\rho}=0$, which in turn depends upon the number of charges in $\mathcal{V}$ and upon $\mathcal{V}$ itself. If $\mathcal{V}=\mathcal{V}(t)$ and $n_e=n_e(t)$, then

$$\rho=\frac{n_e e}{V}\Rightarrow\dot{\rho}=\frac{\delta n_e}{\delta t}\frac{e}{\mathcal{V}}-\frac{n_e e\dot{\mathcal{V}}}{\mathcal{V}^2}=\rho\left(\frac{1}{n_e}\frac{\delta n_e}{\delta t}-\frac{\dot{\mathcal{V}}}{\mathcal{V}}\right)\quad\mathcal{V}=\mathcal{V}(t)\quad n_e=n_e(t): \tag{67}$$

(67) emphasizes that $\dot{\rho}\neq 0$, through which has been defined $\nabla\cdot\mathbf{v}_+$ in (65), requires in general variable volume of space and number n_e of charges; thus the third (66) does not exclude even

$$\nabla\cdot\mathbf{H}'\neq 0\quad\mathcal{V}=\mathcal{V}(t)\quad n_e=n_e(t). \tag{68}$$

The physical meaning of $\dot{\mathcal{V}}\neq 0$ and $\dot{n}_e\neq 0$ along with the consequent (68) will be concerned in the next section, focused precisely on the new field $\mathbf{H}'$.

2) In fact this model introduces contextually the fields $\mathbf{E}$ and $\mathbf{H}$ via the vectors $\mathbf{U}_+$ and $\mathbf{U}_-$ according to (55). Let us show that this feature is not merely formal, *i.e.* $\mathbf{E}+\mathbf{H}$ and $\mathbf{E}-\mathbf{H}$ have actual physical meaning; in effect, once having discarded (60) and (61), the e.m. field is reasonably due to a combination of both fields. Calculate from the second and third (59)

$$\nabla\times\frac{\partial\mathbf{E}}{\partial t}=-\frac{1}{c}\frac{\partial^2\mathbf{H}}{\partial t^2}\quad\nabla\times\frac{\partial\mathbf{H}}{\partial t}=\frac{1}{c}\frac{\partial^2\mathbf{E}}{\partial t^2}+\frac{4\pi}{c}\frac{\partial\mathbf{J}}{\partial t} \tag{69}$$

and put

$$\nabla\times\frac{\partial\mathbf{E}}{\partial t}=-\nabla^2\mathbf{H}c=-\frac{1}{c}\frac{\partial^2\mathbf{H}}{\partial t^2}\quad-\frac{\partial\left(4\pi\mathbf{J}/c-\nabla\times\mathbf{H}\right)}{\partial t}=\nabla^2\mathbf{E}c=\frac{1}{c}\frac{\partial^2\mathbf{E}}{\partial t^2}; \tag{70}$$

then one infers

$$\nabla^2\mathbf{H}=\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\quad\nabla^2\mathbf{E}=\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}, \tag{71}$$

which of course are concurrently obtained and require

$$\nabla\times\frac{\partial\mathbf{E}}{\partial t}=-\nabla^2\mathbf{H}c\quad-\frac{\partial\left(4\pi\mathbf{J}/c-\nabla\times\mathbf{H}\right)}{\partial t}=\nabla^2\mathbf{E}c. \tag{72}$$

Thus the e.m. waves are characterized by their fields both propagating at the same velocity c according to the respective $\mathbf{f}(x+ct)+\mathbf{g}(x-ct)$, along a given coordinate *x*-axis defined in agreement with (37) by the constant unit vector $\mathbf{x}_0$ that identifies the components $H_x=\mathbf{H}\cdot\mathbf{x}_0$ and $E_x=\mathbf{E}\cdot\mathbf{x}_0$ of the fields.

3) The balance between number of unknowns and equations, taking of course the local coordinates x,y,z,t as free input parameters, is:

8 unknown values at any coordinate where the functions (ρ, $\mathbf{v}$, $\mathbf{E}$, $\mathbf{H}$, $\partial\mathbf{E}/\partial t$, $\partial\mathbf{H}/\partial t$, $\partial^2\mathbf{E}/\partial t^2$, $\partial^2\mathbf{H}/\partial t^2$) are calculable.

8 equations *i.e.* 4 in (59) + 2 in (71) + 2 in (72).

Thus also (71), concerning in particular the wavelike propagation of the e.m. field, are admissible in the conceptual frame of the Maxwell equations compliant with the simplifying assumptions (54).

4) As a further corollary of (59) and owing to (58), quote

$$-\frac{\mathbf{v}}{c}\cdot(\nabla\times\mathbf{H})=\nabla\cdot\left(\frac{\mathbf{v}}{c}\times\mathbf{H}\right)=\nabla\cdot\frac{\mathbf{F}_L}{e}=\frac{\mathbf{v}}{c^2}\cdot\frac{\partial\mathbf{E}}{\partial t}+4\pi\frac{\mathbf{v}\cdot\mathbf{J}}{c^2}$$
$$\frac{\mathbf{v}\cdot\mathbf{J}}{c^2}=\rho\frac{v^2}{c^2}\quad \mathbf{F}_L=e\frac{\mathbf{v}}{c}\times\mathbf{H}, \tag{73}$$

being $\mathbf{F}_L$ the Lorentz force. Put now

$$\mathbf{F}_L=\nabla\mathcal{E}_L. \tag{74}$$

Note that not necessarily the energy $\mathcal{E}_L$ must be related to the meaning of potential energy, it is enough to implement the dimensional worth of the proposed definition; then (74) and (73) yield

$$\nabla\cdot\mathbf{F}_L=\nabla^2\mathcal{E}_L=\frac{v}{c^2}\cdot\left(\frac{\partial(e\mathbf{E})}{\partial t}+4\pi(e\mathbf{J})\right)$$
$$=\frac{\mathbf{v}}{c^2}\cdot\left(\frac{\partial\mathcal{F}}{\partial t}+\frac{4\pi e^2}{\mathcal{V}}\mathbf{v}\right)=\frac{\mathbf{v}}{c^2}\cdot\frac{\partial\mathcal{F}}{\partial t}+\frac{4\pi e^2}{\mathcal{V}}\frac{v^2}{c^2}\quad \mathcal{F}=e\mathbf{E}.$$

Multiply both sides by the proportionality factor $(hc)^2/\epsilon$, being ϵ energy; the result reads

$$\frac{(hc)^2\nabla^2\mathcal{E}_L}{\epsilon}=\frac{h^2}{\epsilon}\frac{\mathbf{v}\cdot\partial\mathcal{F}}{\partial t}+\frac{4\pi e^2h^2}{V}\frac{v^2}{\epsilon}, \tag{75}$$

where all terms have physical dimensions of square energy. Owing to (27), v^2 is proportional to $(pc)^2/(\varepsilon/c)^2$, *i.e.* the second addend accounts for the form of square energy proportional to $(pc)^2$. So this result reads

$$\varepsilon^2=\varepsilon'^2+\varepsilon''^2\quad \varepsilon''^2\propto(pc)^2, \tag{76}$$

i.e. it is consistent with the invariant energy equation of the special relativity.

5) The Maxwell equations, as written in (59) and (72)

$$\nabla\cdot(\mathbf{E}\pm\mathbf{H})=4\pi\rho\quad c\nabla\times(\mathbf{E}\pm\mathbf{H})=\frac{\partial(\mathbf{E}\pm\mathbf{H})}{\partial t}+4\pi J$$
$$c\nabla\times\frac{\partial(\mathbf{E}\pm\mathbf{H})}{\partial t}=c^2\nabla^2(\mathbf{E}\pm\mathbf{H})+4\pi\frac{\partial J}{\partial t}, \tag{77}$$

can be merged via

$$\mathcal{S}=\mathbf{E}+\mathbf{H}\quad \mathcal{D}=\mathbf{E}-\mathbf{H}, \tag{78}$$

being of course

$$\frac{\mathcal{S}+\mathcal{D}}{2}=\mathbf{E}\quad \frac{\mathcal{S}-\mathcal{D}}{2}=\mathbf{H}. \tag{79}$$

The electric and magnetic fields combined as new fields $\mathcal{S}$ and $\mathcal{D}$ yield

$$\mathcal{D}\cdot\mathcal{S}=E^2-H^2\quad \mathcal{S}\cdot\mathcal{S}=E^2+H^2+2\mathbf{E}\cdot\mathbf{H}\quad \mathcal{D}\cdot\mathcal{D}=E^2+H^2-2\mathbf{E}\cdot\mathbf{H} \tag{80}$$

while being in principle

$$\mathbf{E}\cdot\mathbf{H}=0\quad \text{or}\quad \mathbf{E}\cdot\mathbf{H}\lesseqgtr 0; \tag{81}$$

if in particular $\mathbf{E}$ and $\mathbf{H}$ are orthogonal, e.g. an e.m. wave in the vacuum, then

$$\mathcal{D} \times \mathcal{S} = (\mathbf{E} - \mathbf{H}) \times (\mathbf{E} + \mathbf{H}) = 2\mathbf{E} \times \mathbf{H}. \tag{82}$$

Also:

- $\mathcal{S}$ and $\mathcal{D}$ fulfill the D'Alembert wave equation, as it appears summing or subtracting (71) side by side;
- If $\mathbf{E}$ and $\mathbf{H}$ are orthogonal then $\mathcal{S}^2 \equiv \mathcal{D}^2$ represents a free e.m. wave propagating along the direction of $\mathbf{v}$.
- The propagation of the e.m. wave does not imply the presence of free charges, *i.e.* $\mathbf{E}$ and $\mathbf{H}$ are intrinsic properties of the wave.
- On the one hand (80) to (82) show that $\mathcal{S}$ and $\mathcal{D}$ have actual physical meaning: $\mathcal{D} \times \mathcal{S}$ implies the Poynting vector, calculable via (59) as it is known, $\mathcal{D} \cdot \mathcal{S}$ the Lagrangian density of a free field.
- On the other hand the coefficient 2 in (80) and the fact that $2\mathbf{E} \cdot \mathbf{H}$ does not appear in $\mathcal{S} \cdot \mathcal{D}$ suggest regarding (80) as Lagrangian and Hamiltonian densities:

$$\begin{aligned} \mathcal{L}_{\mathcal{V}} &= (T_{\mathcal{V}} \pm \mathbf{E} \cdot \mathbf{H}) - (U_{\mathcal{V}} \pm \mathbf{E} \cdot \mathbf{H}) = T_{\mathcal{V}} - U_{\mathcal{V}} \\ \mathcal{H}_{\mathcal{V}} &= (T_{\mathcal{V}} \pm \mathbf{E} \cdot \mathbf{H}) + (U_{\mathcal{V}} \pm \mathbf{E} \cdot \mathbf{H}) = T_{\mathcal{V}} + U_{\mathcal{V}} \pm 2\mathbf{E} \cdot \mathbf{H}; \end{aligned} \tag{83}$$

i.e., whatever $T_{\mathcal{V}}$ and $U_{\mathcal{V}}$ might be, both T and U actually include the scalar $\mathbf{E} \cdot \mathbf{H}$, which in turn accounts for the presence of free charges in the space time with $\mathbf{E} \cdot \mathbf{H} \neq 0$. Indeed merge both chances (81) to write

$$\mathbf{H} = \mathbf{H}_w + \mathbf{H}_{fc} \quad \mathbf{E} = \mathbf{E}_w + \mathbf{E}_{fc} \quad \mathbf{E}_w \cdot \mathbf{H}_w = 0, \tag{84}$$

where the subscripts stand for *wave* and *free charge*. Owing to the last position, the orthogonal character of the $\mathbf{E}_w$ and $\mathbf{H}_w$ fields is still definable for any e.m. wave regardless of the possible presence of free charges. In other words, these positions are compatible with both inequalities (81) while being also compliant with the properties of the e.m. wave itself, which results consisting of two orthogonal fields propagating through the vacuum or a matter medium: "matter" is by definition everything allowing and requiring $|\mathbf{v}| < c$.

In principle, therefore, neither new terms nor additional hypotheses are necessary to introduce explicitly in (80) free charges in the space time through which propagates the e.m. wave characterized by its own $\mathbf{E}_w$ and $\mathbf{H}_w$. Thus (78), which imply (71) and (84), are not "ad hoc" hypotheses, rather they aim to plug the fields into the frame of a propagating e.m. wave. At this point, specify also $\mathbf{v}$ defining $\mathbf{J}$ in (59) as

$$\mathbf{v} = \mathbf{v}_w + \mathbf{v}_{fc} \tag{85}$$

with analogous meaning of symbols; so $\mathbf{v}_w$ is the velocity of the e.m. wave, whereas $\mathbf{v}_{fc}$ is the average velocity of the charges possibly present in the environment where travels the wave. Clearly it is convenient to define $\mathbf{v}_w$ normal to $\mathbf{E}_w$ and $\mathbf{H}_w$ in order that these three vectors define effectively the fields of an e.m. wave and its propagation direction. Put now

$$\mathbf{E} \cdot \mathbf{H} = (\mathbf{H}_w \cdot \mathbf{E}_{fc} + \mathbf{E}_w \cdot \mathbf{H}_{fc}) + \mathbf{E}_{fc} \cdot \mathbf{H}_{fc} \quad \mathbf{E}_w \cdot \mathbf{H}_w = 0 \quad \mathbf{E}_w \cdot \mathbf{v}_w = \mathbf{H}_w \cdot \mathbf{v}_w = 0: \tag{86}$$

once having already found (59), the physical meaning of (86) is intuitively understood according to the Maxwell equations. Reasonably (84) and (85) concern and account for the quantum fields (54). Analogous considerations hold in principle rewriting (84) as $\mathbf{H}=\mathbf{H}_w+\mathbf{H}_{w'}$ and $\mathbf{E}=\mathbf{E}_w+\mathbf{E}'_w$, with $\mathbf{E}\cdot\mathbf{H}$ describing now photon-photon interaction through $\mathbf{E}_w\cdot\mathbf{H}_{w'}$ and $\mathbf{E}_{w'}\cdot\mathbf{H}_w$ of e.m. waves propagating along $\mathbf{v}_w$ and $\mathbf{v}'_w$.

4. Corollary of the Model

Equation (68) outlines the possible existence of magnetic monopoles, thought of as isolated north and south poles of ordinary magnets [3]. If magnetic monopoles floating independently each other as separate magnetic charges actually existed, continuity equation for monopole currents should be also definable. Alternatively regard the monopoles as mere quantum energy fluctuations randomly forming and annihilating in $\mathcal{V}$, *i.e.* a virtual cloud instead of a real stream of particles, due to the interaction between ordinary nanosized magnets and quantum vacuum. To explain this idea rewrite first $\dot{\rho}$ of (35) to (68) with explicit notation

$$\rho_{H'}=\frac{q_{H'}}{\mathcal{V}}\quad \mathbf{J}_{H'}=\rho_{H'}\mathbf{v}_{H'}\quad \mathbf{v}_{H'}\equiv\mathbf{v}_+\quad \dot{\rho}_{H'}=\frac{\dot{q}_{H'}}{\mathcal{V}}-\rho_{H'}\frac{\dot{\mathcal{V}}}{\mathcal{V}}, \tag{87}$$

being $q_{H'}$ the amount of virtual magnetic charges that displace randomly in $\mathcal{V}$ at the average velocity $\mathbf{v}_+$ of (35). Comparing (35) and (87),

$$2\frac{\dot{\rho}_{H'}}{\rho_{H'}}=\nabla\cdot\mathbf{v}_{H'}-\frac{\sigma}{\rho_{H'}}\ \Leftrightarrow\ 2\frac{\dot{\rho}_{H'}}{\rho_{H'}}=2\frac{\dot{q}_{H'}}{q_{H'}}-2\frac{\dot{\mathcal{V}}}{\mathcal{V}}, \tag{88}$$

one infers that

$$\nabla\cdot\mathbf{v}_{H'}\ \Leftrightarrow\ 2\frac{\dot{q}_{H'}}{q_{H'}} \tag{89}$$

whereas the non-conservation term σ of (35) corresponds to

$$\frac{\sigma}{\rho_{H'}}\ \Leftrightarrow\ 2h\frac{\dot{\mathcal{V}}}{\mathcal{V}}. \tag{90}$$

Owing to (68) $\nabla\cdot\mathbf{H}'\neq 0$ is justified by $\dot{\mathcal{V}}\neq 0$ and thus by $\sigma\neq 0$, which in turn skips the continuity Equation (34). The physical meaning of (90) agrees with the idea of creation and annihilation of couples of virtual magnetic monopoles in a resonant system *quantum vacuum* ↔ *nanosized magnet* in $\mathcal{V}$, coherently with the factor 2, whereas $\mathbf{J}_{H'}$ of (87) describes the displacement of separate virtual magnetic charges $q_{H'}$ non-conserved by definition due to their transient lifetime. In general to release free particles from a bound system are necessary splitting energy plus additional energy to give the split particles the necessary kinetic energy to escape independently each other. To explain this point write according to (1)

$$2h\frac{\dot{\mathcal{V}}}{\mathcal{V}}=\epsilon_{H'_+}+\epsilon_{H'_-}+\epsilon_0\quad \delta\epsilon_{H'}=2h\frac{\dot{\mathcal{V}}}{\mathcal{V}}-\epsilon_0=n\hbar\quad \epsilon_{vac}=\mathcal{V}\eta_{vac}. \tag{91}$$

Let the driving energy to form free monopoles be $2h\dot{\mathcal{V}}/\mathcal{V}$ of (90); it stems from the time evolution of quantum vacuum, e.g. its expansion rate per unit volume. Moreover let ϵ_0 be the binding energy of the monopoles in a standard nanosized magnet and $\epsilon_{H'_+}+\epsilon_{H'_-}$ the kinetic energies of free magnetic monopoles. Thus the magnetic charges already existing in their bound state split into couples of separate free particles in a variable volume $\mathcal{V}+\dot{\mathcal{V}}\delta t=\mathcal{V}+\dot{\mathcal{V}}n\hbar/\delta\epsilon_{H'}$, provided that $\delta\epsilon_{H'}>0$ accounts for the kinetic energy of the monopoles activated by the splitting process. This model reminds the idea of light driven photoelectric effect in solids to introduce an analogous quantum vacuum fluctuation driven "nanomagnet-vacuum" interaction: the splitting energy ϵ_0 plays the threshold role analogous to the electron work function, whereas $2h\dot{\mathcal{V}}/\mathcal{V}$ and $\delta\epsilon_{H'}$ replace $h\nu$ and $\delta\epsilon_{kin}$ of (8). So (91) yield two equations

$$2h\frac{\dot{\mathcal{V}}}{\mathcal{V}}=\left(\epsilon_{H'_+}+\frac{\epsilon_{vac}}{2}\right)+\left(\epsilon_{H'_-}+\frac{\epsilon_{vac}}{2}\right)$$

$$h\frac{\dot{\mathcal{V}}}{\mathcal{V}}-\frac{\epsilon_{vac}}{2}=\frac{\delta\epsilon_{H'}}{2}\begin{array}{l}\nearrow\ \dfrac{\epsilon_{H'_+}+\epsilon_{H'_-}}{2}=monopole\ kinetic\ energy\\ \searrow\ \epsilon_{vac}/2=vacuum\ energy\ per\ monopole\end{array}\tag{92}$$

both involving $\epsilon_{vac}/2$. The first equation, energy balance of monopole-quantum vacuum interaction, emphasizes that ϵ_{vac} is shared between both magnetic charges; the second equation required by (1) reads

$$2\left(h\frac{\dot{\mathcal{V}}}{\mathcal{V}}-\frac{\epsilon_{vac}}{2}\right)\delta t=n\hbar\ \Rightarrow\ \delta\epsilon_{H'}=\frac{n}{2}\frac{\hbar}{\delta t}\quad h\frac{\dot{\mathcal{V}}}{\mathcal{V}}-\frac{\epsilon_{vac}}{2}=\delta\epsilon_{H'}.\tag{93}$$

So $V=0$ highligths that ϵ_{vac} in (92) balances binding energy of $q_{H'_+}$ and $q_{H'_-}$, while $2h\dot{\mathcal{V}}/\mathcal{V}$ provides the additional energy to exceed ϵ_{vac} and allows the kinetic energies $\epsilon_{H'_+}$ and $\epsilon_{H'_-}$ inherent $\mathbf{v}_{H'}$ of $\mathbf{J}_{H'}$ of (87). In short, according to (92) the key property that triggers the interaction is actually the zero point energy of the quantum vacuum per monopole, *i.e.* ϵ_{vac} rises the whole nanosized magnet to its upper limit of stability, whereas the further vacuum fluctuation energy provides both monopoles with kinetic energy.

- On the one hand this mechanism requires $\mathcal{V}$ such that its corresponding enclosed energy fulfills the threshold energy necessary to create at least one couple of monopoles: the smaller η_{vac}, the greater $\mathcal{V}$ corresponding to $\epsilon_{vac}=\mathcal{V}\eta_{vac}$ of the third (91).
- On the other hand the energy balance of the splitting mechanism should fit the form of the first (92). Note the the sequence of possible *n*/2 in (93) reads 1/2, 1, 3/2, 2, … *i.e.* 1/2, 1, 1 + 1/2, 2, …: so, whatever in general the arbitrary *n* might be, the sequence of allowed energy states consists of arbitrary integers to each one of which is summed its own zero point term *n* + 1/2. In effect, with quantized $\epsilon_{H'_+}$ and $\epsilon_{H'_-}$, this is the form (92) of both monopoles once regarding ε_0 as quantum vacuum zero point energy: this confirms that the upper limit of stability of the nanosized magnet interacting with the quantum

vacuum concerns the vacuum zero point energy, whose fluctuations merely govern the kinetic energies of the escaped monopoles during their lifetime δt. In this threshold model all allowed energies $n\hbar\dot{\mathcal{V}}/\mathcal{V}$ include the zero point energy, the left hand side of (93) fulfills $\delta\epsilon_{H'}\delta t \ge \hbar/2$.

The order of magnitude of ϵ_0 is estimated via (91). To evaluate η_{vac} note that owing to (26) and (27)

$$\frac{\eta_{vac}}{c^2} = \frac{\epsilon_{vac}/c^2}{\mathcal{V}} = \frac{p_{vac}}{\mathcal{V}v} = \frac{J_{vac}}{v} = \rho_{vac}, \tag{94}$$

which defines $\eta_{vac} = \rho_{vac}c^2$ via $J_{vac} = \rho_{vac}v$, as reasonably expected. These values are calculated from cosmological data in [4]: ρ_{vac} and η_{vac} are equal to $(60.3 \pm 1.3) \times 10^{-31}$ g/cm^3 and 5.4×10^{-9} erg/cm^3. Here it is sketched how to find these values starting from (2) in the conceptual frame hitherto exposed.

By dimensional reasons $h\left(\hbar G/c^2\right) = energy \times length^3$; thus, being $\eta = energy/length^3$, write in general

$$\frac{\hbar^2 2\pi G/c^2}{\eta} = \ell^6 \qquad \frac{\hbar^2 2\pi G}{c^2}\eta = \epsilon^2, \tag{95}$$

where ε and ℓ symbolize the characteristic energy and length of the splitting process. The resulting η is

$$\eta = \left(\frac{\epsilon c}{\hbar}\right)^2 (2\pi G)^{-1} = \left(\frac{c}{\tau}\right)^2 (2\pi G)^{-1} \qquad \tau = \frac{\hbar}{\epsilon}.$$

In this result appear only fundamental constants of nature and the time constant τ, now defined to give η the specific physical meaning of vacuum energy density. A straightforward way to express η entirely as a function of cosmological data is to replace τ^{-1} with today's value of the Hubble factor H_u, which actually has physical dimensions $time^{-1}$. In fact the universe expansion has been previously mentioned to exemplify a possible chance of justifying $\dot{\mathcal{V}}/\mathcal{V}$ in (90); this preliminary idea is now implemented in (95) to evaluate numerically the vacuum energy density η_{vac} of universe. Thus

$$\eta = \frac{(H_u c)^2}{2\pi G} \qquad H_u = 2.2 \times 10^{-18}\,\text{s}^{-1} \tag{96}$$

yields, replacing in (95),

$$\epsilon^2 = \hbar^2 H_u^2 \qquad \epsilon = \pm\hbar H_u = \frac{h\dot{\mathcal{V}}}{\mathcal{V}}$$

suggests that merging (93) and (95) to calculate $\epsilon_0/2$, one finds

$$\frac{\epsilon}{2} = \frac{h\dot{\mathcal{V}}}{\mathcal{V}} = \frac{1}{2}\sqrt{\frac{\hbar^2 2\pi G}{c^2}\eta_{vac}} \qquad \ell = \left(\frac{\hbar^2 2\pi G}{c^2 \eta_{vac}}\right)^{\frac{1}{6}} \qquad \eta_{vac} = \frac{1}{2}\frac{(H_u c)^2}{2\pi G} \qquad \rho_{vac} = \frac{\eta_{vac}}{c^2}; \tag{97}$$

i.e. the vacuum energy density is the zero point energy per unit volume of quantum vacuum fluctuations (93) $n\hbar\dot{\mathcal{V}}/\mathcal{V}$ triggered by the dynamical expansion energy of universe hH_u. The numerical values are

$$\eta_{vac} = 5.2 \times 10^{-9}\,\frac{\text{erg}}{\text{cm}^3} \qquad \rho_{vac} = 58 \times 10^{-31}\,\frac{\text{g}}{\text{cm}^3}$$
$$\ell = 6.8 \times 10^{-13}\,\text{cm} \qquad \epsilon = 8.2 \times 10^{-46}\,\text{erg}. \tag{98}$$

The good agreement of η_{vac} and ρ_{vac} with the values [4] supports (96) and (97), while being

$$\frac{\dot{\mathcal{V}}}{\mathcal{V}} = 1.2\times10^{-19}\ \text{s}^{-1}. \tag{99}$$

To implement (98) introduce the physical features of $\mathcal{V}$. Define $V \propto e^2/\hbar$. Indeed $e^2/\hbar$ has physical dimensions of velocity, so that any velocity can be in principle expressed as $\kappa e^2/\hbar$. As by definition $\mathcal{V}$ is the volume of quantum vacuum whose fluctuation allows splitting the monopoles, then: the greater their average velocity as soon as they form, the greater the volume allowing in fact their own delocalization. In other words the condition consistent with finite life time of monopoles flying independently each other reads

$$\mathcal{V} = \kappa\frac{e^2}{\hbar} \quad \frac{e^2}{\hbar} = 2.2\times10^{8}\ \text{cm/s}, \tag{100}$$

being κ a dimensionless proportionality coefficient. If the reasoning is correct, κ of (100) should be of the order of unity; in general, a proportionality constant significantly different from 1 means that some hidden effect is missing in the proposed reasoning. So $\kappa = 1$ for an order of magnitude estimate of $\mathcal{V}$ yields

$$\mathcal{V} \approx 2.2\times10^{8}\ \text{cm}^3 \quad \dot{\mathcal{V}} \approx 2.7\times10^{-11}\ \text{cm}^3\cdot\text{s}^{-1}; \tag{101}$$

this is the volume where is delocalized the nano-sized magnet with upper threshold energy ϵ_{vac} along with the possible ϵ_{H_+} and ϵ_{H_-}, both with their own kinetic energy triggered by the n-th vacuum fluctuation. So, for a couple of magnetic monopoles,

$$\epsilon_0 = \eta_{vac}\mathcal{V} \approx 5.3\times10^{-9}\times2\times10^{8} \approx 1.1\,\text{erg} \Rightarrow \epsilon_0 \sim 687\ \text{GeV} \quad \delta t \sim 10^{-27}\ \text{s}; \tag{102}$$

thus each monopole should require a threshold energy ~340 GeV to be formed. Eventually, as concerns ϵ of (98), note that comparing $h\mathcal{V}$ and $\epsilon\mathcal{V}$ one finds consistent values ~1.8 × 10^{-37} erg·cm^3. The fact that $\epsilon\mathcal{V} = h\mathcal{V}$ suggests regarding $\epsilon\mathcal{V}$ as a physical property of the quantum vacuum-nanosized magnet interaction in the aforesaid vacuum-magnetic interaction, reminiscent of the analogous Fermi constant of the weak interaction.

As concerns the magnetic charges of (87) and (91), the quantized result obtained by Dirac in 1931 reads The analytical form of the equation that introduces the magnetic charge $q_{H'}$ of the monopole, either $q_{H'_+}$ or $q_{H'_-}$, reminds (1). The quantization of the electric charge has been inferred in (23) and (24); thus (103) is reasonably related to these equations, while both $q_{H'}$ refer to the field $\mathbf{H}'$ of (66). Read (23) as

$$\pm q^2\frac{\delta r\delta\ell}{r^2} = n\hbar c \quad \delta\ell = c\delta t, \tag{103}$$

which yields

$$2q_{H'}q = \mp n\hbar c \quad 2q_{H'} = q\frac{\delta r\delta\ell}{r^2} \quad 2q_{H'} = q_{H'_+} + q_{H'_-}: \tag{104}$$

the first position is mere rewriting of the given definition of $q_{H'}$, coherent with (91) in turn related to (88), the third position reminds that (91) requires 2 magnetic charges contextually involved from the splitting of one nano-sized magnet. The second position is the key definition; it implies

$$2q_{H'} = \pm\frac{n\hbar c}{q} \quad 2q_{H'} = \left(\frac{q}{r^3}\delta\mathbf{r}\right)\cdot(r\delta\ell) = \delta\mathbf{E}\cdot\delta\mathbf{S}$$
$$\delta\mathbf{S} = r\delta\ell = r\delta\ell\mathbf{u} \quad \delta\mathbf{E} = \frac{q}{r^3}\delta\mathbf{r}, \tag{105}$$

being $\mathbf{u}$ unit vector normal to $\delta\mathbf{S}$. So, with the notation (8), $\delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0$ and $r\delta\ell = r\ell_1 - r\ell_0$ define $\delta\mathbf{E}$ and $\delta\mathbf{S}$. This reasonable conclusion of (104) confirms that the first (105) is the Dirac result once specifying $q \equiv e$.

5. Discussion

As stated in the Section 1, the Maxwell equations are the main result among many outcomes obtainable through the present model: e.g. (43), (8) and (45) are also obtained as a byproduct of (1). This approach configures the model into a broad framework, purposely aimed to emphasize the link between the Maxwell equations and fundamental laws of physics.

The chance of plugging (59) in a broad context of physical information is likely more significant than the initial motivation alone. An example is the link between vacuum energy density (95) consequent to (2) and monopole formation mechanism, which however must be experimentally confirmed at the indicated energy. Despite the classical character of the Section 2, have been obtained through the uncertainty the Equations (118) and (27) of the special relativity along with the successive (76) without additional hypotheses.

The quantum basis is coherent with the corpuscle/wave quantum properties of matter inherent (1): defining conjugate δx and δp_x implies that the random delocalization of a corpuscle in δx and its wave behavior inherent (7) and (6) along with (4) itself are aspects of matter behavior conceptually correlated. Actually Equations (1) overcome the quantum duality wave/corpuscle by accounting straightforwardly for both: (4), (9) and (18) imply the wave behavior of light, including the quantization driven photoelectric effect of (8) as well, whereas (44) and (45) concern corpuscles of matter. In fact the wave equation is explictly inferred itself. Divide both sides of (1) by x and define

$$\frac{\hbar}{x+x_0} = \psi p_x \quad \frac{\delta x}{x+x_0} = \delta\psi \quad x>0 \quad \psi = \psi(x, p_x),$$

being x_0 a constant coordinate and ψ a function to be found. The physical dimensions of these definitions are consistent. Assume for simplicity $x \gg x_0$ and then divide side by side the resulting equations; it yields

$$\frac{\hbar}{x} = \psi p_x \quad \frac{\delta x}{x} = \delta\psi \Rightarrow \frac{\delta x}{\hbar} = \frac{n}{\delta p_x} = \frac{\delta\psi/\psi}{p_x}. \tag{106}$$

The result at the right hand side shows that replacing $\delta p_x \delta x$ with $p_x \delta x$

implies replacing n with $\delta\psi/\psi$, which therefore has analogous physical meaning: as p_x alone cannot define a range of allowed quantum states concurring to the total n, the conclusion is that now likewise to (4) $\delta\psi = n\psi$ indicates generically $\psi_{k_{\min}} \le \psi_k \le \psi_{k_{\max}}$ for $n_{k_{\min}} \le n_k \le n_{k_{\max}}$. With the definitions (106), (1) turns into

$$n\psi\, p_x = \delta\psi\delta p_x = n\hbar\frac{\delta\psi}{\delta x} \Rightarrow \frac{1}{\hbar}\psi\, p_x = \frac{\delta\psi}{\delta x},$$

whereas by consequence

$$\delta\log(\psi) = \frac{p_x\delta x}{\hbar} \quad \text{i.e.} \quad \frac{\psi}{\psi_0} = \exp\left(\frac{p_x\delta x}{\hbar}\right). \tag{107}$$

Despite in this way the function ψ diverges for $\delta x \to \infty$ with finite p_x, it is enough to multiply both sides of the first (107) by i; *i.e.*, replacing $p_x \to iP_x$ and $\psi \to i\psi$, one obtains

$$\frac{i}{\hbar}\psi^* P_x = \frac{\delta\psi^*}{\delta x} \quad \text{along with} \quad \frac{\psi^*}{\psi_0} = \exp\left(\frac{iP_x\delta x}{\hbar}\right) \tag{108}$$

with ψ^* complex and P_x real, while ψ_0 is normalization constant of $\psi^*\psi$. Analogous reasoning holds for the classical energy wave equation.

Today the quantum theory is implemented prevalently via its wave formulation [5]; nevertheless this is clearly reductive. The way to calculate the energy levels of hydrogenlike and many electron atoms is shown in [1]. Here is sketched for completeness how (1) regards this problem via (9). Write with the help of (1)

$$\frac{\hbar G}{c^2} = \frac{mG}{nc^2}\frac{\delta r^2}{\delta t} \quad v = \frac{\delta r}{\delta t} \Rightarrow (n\hbar)^2 = m\delta r^2\delta E \quad \delta t = \frac{n\hbar}{\delta E} \quad \delta r^2 = \delta\mathbf{r}\cdot\delta\mathbf{r}.$$

Specify δE as energy range allowed to an e.m. system of charges δr apart, *i.e.* $\delta E = Ze^2/\delta r$. Replacing this condition of classical Coulomb approximation in the equation of δE, one finds $(n\hbar)^2/Zme^2 = \delta r$ that in turn yields $\delta E = Z^2e^4m/(n\hbar)^2$. Therefore with the notation of (8)

$$\delta E = E_1 - E_0 = \frac{Z^2e^4m}{(n\hbar)^2} \quad \delta r = r_1 - r_0 = \frac{(n\hbar)^2}{Zme^2} \Rightarrow \delta E\delta r = Ze^2. \tag{109}$$

Follow now the reasoning carried out in (30) to correlate $E_1 - E_0$ and $r_1 - r_0$, *i.e.* either: $E_1 \leftrightarrow r_1$ and $E_0 \leftrightarrow r_0$ or $-E_1 \leftrightarrow r_0$ and $-E_0 \leftrightarrow r_1$. The latter case is more interesting because it implies

$$-\frac{Z^2e^4m}{(n_1\hbar)^2} \leftrightarrow \frac{(n_0\hbar)^2}{Zme^2} \quad -\frac{Z^2e^4m}{(n_0\hbar)^2} \leftrightarrow \frac{(n_1\hbar)^2}{Zme^2}:$$

so $\delta r_1 = (n_1\hbar)^2/Zme^2$ is related to the negative energy $-E_0$; the same holds for $\delta r_0 = (n_0\hbar)^2/Zme^2$ related to $-E_1$. The opposite would clearly be true relating δr_1 and δr_0 to the respective positive energies E_1 and E_0. The former case is interesting as it concerns binding e.m. interaction between opposite charges. Actually n_1 and n_0 do not define different numbers of allowed quantum states, because n symbolizes by definition any integer as stated in (3). This way to account for both $\pm e^2/\delta r$ is relevant: $(n\hbar)^2/Zme^2$ yields clearly the sequence of Bohr radii, whatever the notation of *n* might be, whereas $Z^2e^4m/(n\hbar)^2$ is twice the Bohr energy level whatever the notation of *n* might be. Rewrite now the last (109) as $(\delta E/2)2\delta r$: this model implements uncertainty ranges, not deterministic positions and distances. Thus an electron at a radial distance r_{Bohr} from the nucleus must be regarded as an electron delocalized in a diametral uncertainty range $2r_{Bohr}$, whence the idea of defining $\delta E/2$ by consequence as the actual Bohr energy, *i.e.*

$$r_{Bohr}=\frac{(n_0\hbar)^2}{Zme^2}\quad E_{Bohr}=-\frac{Z^2e^4m}{2(n\hbar)^2}. \tag{110}$$

According to (45) and (46) *n* concerns a statistical set of many particles, whereas (110) show that in fact *n* counts different quantum states of a single particle. Hence $\delta\varepsilon$ of (1) includes in principle all ε_i defining the whole energy distribution of several atoms in a solid body or the progressive energy levels ε_i of a unique particle, e.g. one electron in hydrogenlike atoms. In other words it means regarding *n* either as the total number of states allowed to all *i*-th particles in a set or all *i*-th energy levels of one particle only: in the former case $\Sigma n_i=n$ implies arbitrary integers $n_i\neq 1$ respectively pertaining to the different particles of a body, whereas in the latter case $n_i\equiv 1$ by definition, being however still true that $\Sigma n_i=n$ once summing *n* times the occupancy of a unique electron over all its possible energy levels. The significant fact is that anyway *n* counts two possible occupancy ways of allowed states in a physical system, by means of various $n_i\neq 1$ of different particles or different quantum states of a unique particle, thus with same $n_i=0$ or $n_i=1$ depending on which level is actually occupied. This implies in turn that the signs of (45) are actually related to either filling mode of allowed quantum states: even regardless of the spin of particles, involved by reasons concerned in [6], simple considerations show that the distribution (45) with the minus sign only allows to condensate all particles in a unique ground level under appropriate physical conditions.

Moreover follows now an example of information obtainable merging corpuscular and wave information, *i.e.*

$$p=mv=m\lambda\nu=m\frac{h}{p}\nu\ \Rightarrow\ \frac{p^2}{m}=h\nu=2T;$$

owing to (1) and (4) $h\nu$ reads

$$2T=\frac{\hbar}{n\delta t}\ \Rightarrow\ 2T+\frac{\delta U}{n}=0\quad \delta U=-\frac{\hbar}{\delta t}, \tag{111}$$

having defined

$$\delta U=-\frac{\delta r\delta p_r}{n_r\delta t}=-\frac{F_r\delta r}{n_r}\quad F_r=\frac{\delta p_r}{\delta t}.$$

As in fact δU is energy range, anyway it must fulfill $n_r\hbar=\delta U\delta t$. So (111) is the classical virial theorem. If for example $U=-n\left(const/\delta r^k\right)$, then by differentiating

$$\delta U = k\frac{const}{\delta r^{k+1}}\delta r \;\Rightarrow\; 2T = k\delta U.$$

Once having defined U via quantum uncertainty, is self-evident the idea of concerning in (111) average values of all local dynamical variables enclosed in their own ranges.

Intuitively the corpuscular implications of (1), already emphasized, bring straightforwardly to the relativity. Precisely for this reason the present model introduces further information necessary to bridge quantum world and relativity, while configuring in (25) to (35) the framework leading in particular to (8) and classical (59).

To emphasize once more the significance of (2), consider $r_m = mG/c^2$ of (9) to calculate Ac implementing the surface area A defined in (37). As in fact $Ac = length^3/time$, compare this definition with the dimensionless Beckenstein-Hawking entropy S_{BH} via (2); *i.e.*

$$\frac{4\pi r_m^2 c}{\hbar G/c^2} = \frac{4\pi\left(mG/c^2\right)^2 c}{\hbar G/c^2} = \frac{4\pi m^2 G}{\hbar c}$$

$$\Leftrightarrow\; \frac{A_{bh}}{4\ell_{Pl}^2} = \frac{4\pi\ell_{bh}^2}{4\ell_{Pl}^2} = \frac{4\pi\left(2mG/c^2\right)^2}{4\hbar G/c^3} = \frac{4\pi m^2 G}{\hbar c}.$$

The dimensionless ratio at the left hand side is precisely S_{BH}. It shows that S_{BH} is actually a property of the definition (2) of space time: it has been found without introducing preliminarily the black hole radius, which however has been already introduced in (17) via the quantum number of allowed states $n = 2$ for the amount m of matter and reasonably appears here.

It is not surprising that the uncertainty, and thus the Maxwell equations themselves, are compliant with the special relativity. A further consideration appears appropriate in this regard: (1) imply $\delta\varepsilon - v_x\delta p_x = 0$, which in turn for any $v_x^* \neq v_x$ takes the explicit forms

$$\delta\varepsilon_+ + v_x^*\delta p_x = f' \quad \delta\varepsilon_- - v_x^*\delta p_x = f''.$$

The new variable v_x^* can be indeed added or subtracted, while no hypothesis is necessary about the resulting functions f' and f''. Indeed, multiplying side by side these equations, one finds

$$\delta\varepsilon^2 - v_x^{*2}\delta p_x^2 = f'f'' \quad \delta\varepsilon^2 = \delta\varepsilon_+\delta\varepsilon_-, \tag{112}$$

which in turn owing to (1) yields also

$$\frac{(n\hbar)^2}{\delta t^2} - v_x^{*2}\frac{(n\hbar)^2}{\delta x^2} = f'f'' \;\Rightarrow\; \frac{\delta x^2 - \left(v_x^*\delta t\right)^2}{(\delta x\delta t)^2} = \frac{f'f''}{(n\hbar)^2}. \tag{113}$$

The results (112) and (113) merge then into

$$\frac{\delta x^2 - v^{*2}\delta t^2}{(\delta x\delta t)^2} = \frac{\delta\varepsilon^2 - v^{*2}\delta p_x^2}{(n\hbar)^2}. \tag{114}$$

A possible way to regard the denominators of (114) is to correlate precisely $\delta x\delta t$ and $n\hbar$. Thus $\delta x\delta t \propto n\hbar$ implies $\delta x^2 - v_x^{*2}\delta t^2 \propto \delta\varepsilon^2 - v_x^{*2}\delta p_x^2$; trivial considerations on the proportionality constants allow writing

$$\delta x^2 - v_x^{*2}\delta t^2 = \delta x_0^2 - v_0^{*2}\delta t_0^2 \quad \frac{\delta x\delta t}{n\hbar} = \frac{\delta x_0\delta t_0}{n_0\hbar} \quad \delta\varepsilon^2 - v_x^{*2}\delta p_x^2 = \delta\varepsilon_0^2 - v_0^{*2}\delta p_0^2. \quad (115)$$

Let δx and δt be defined in a reference system *R*, whereas the corresponding labeled with 0 in a reference system R_0. Boundary condition: for $\delta x \equiv \delta x_0$ and $\delta t \equiv \delta t_0$, hold for *n* the previous remarks. Then $\delta x\delta t$ must be an invariant in different reference systems. Thus the same must be true for the numerator of (114), *i.e.* v^* must be constant; then also the numerator is an invariant, *i.e.* $v^* \equiv c$.

Equations (114) and (115) merely rewrite (1). Now exemplify how to extend further the implications of (1).

2) Implement (11) rewritten in two ways formally equivalent

$$mc^2 = nh\nu \Rightarrow mc^2 = (n+1)h\nu - h\nu \quad mc^2 = (n-1)h\nu + h\nu. \quad (116)$$

Reminding that $h\nu = hc/\lambda = pc$, as done in (11), the last equations read

$$mc^2 = (n+1)h\nu - pc \quad mc^2 = (n-1)h\nu + pc.$$

A possible way to merge these two results is to multiply them side by side; so trivial manipulations yield

$$\varepsilon^2 = (pc)^2 + (mc^2)^2 - \xi(pc)^2\varepsilon \quad \xi = \frac{2pc - h\nu}{(pc)^2 n} \quad \varepsilon = nh\nu \quad (117)$$

So $\xi \to 0$ for $2pc \approx h\nu$ and/or for $n \to \infty$, in which case (117) reduces to the familiar energy equation of special relativity. I general, however, (117) includes a small correction to the standard energy Equation (14); ξ is defined by terms that decrease with the shared *n*, whereas ε at the left hand side increases with *n*. So ξ becomes more and more negligible for large numbers of states. Note that ξ has physical dimensions of reciprocal energy. Regard thus more expressively $\xi = 2\epsilon^{-1}$, which means $\xi = \xi(\epsilon^{-1})$. The series expansion of ξ yields in general a zero order constant term ξ_0 plus higher order terms $a_k\epsilon^{-k}$ with coefficients a_k; however, since the correction term is expected to be small itself, then it is possible to write (117) as $-\xi_0(pc)^2\varepsilon$. Yet with this correction, compatible with (14) via appropriate reasoning about ζ previously omitted for brevity, (117) is known equation of quantum gravity that solves three cosmological paradoxes [7]; no new hypthesis is necessary to obtain this result.

1) Define by dimensional reasons the energy

$$\varepsilon^2 = \theta\frac{e^4a^2}{c^4} \quad (118)$$

being a^2 square proper acceleration of a charged particle in the vacuum and θ proportionality constant. Implement (117); neglecting for simplicity and brevity the small correction term putting $\xi = 0$, one finds

$$\delta(\varepsilon^2)=\delta\left((pc)^2+(mc^2)^2\right)=\theta\frac{e^2}{c}\delta\left(\frac{e^2a^2}{c^3}\right)\Rightarrow\delta\left((pc)^2\frac{c}{e^2}\right)=\delta\left(\theta\frac{e^2a^2}{c^3}\right).$$

Thus

$$\delta W=\delta\left(\theta\frac{e^2a^2}{c^3}\right)\Rightarrow W=\theta\frac{e^2a^2}{c^3}+const\quad W=(pc)^2\frac{c}{e^2}\quad\theta=\frac{2}{3},\tag{119}$$

being W power by dimensional reasons. With $const=0$ and the given value of θ the result yields the Lorentz invariant power dissipated in an arbitrary volume of space $\mathcal{V}$ by an accelerating charge *e*.

The proportionality coefficient $\theta=2/3$ concerns the case of radiation back reflected at the boundaries of an arbitrary ideal $\mathcal{V}$ enclosing *e*. To justify this factor, find the relationship between pressure P and energy density η. Define the volume $\mathcal{V}$ as $\mathcal{V}=x_0^{3-k}x^k$, where x_0 is an arbitrary constant length and x an arbitrary variable length. As $1\le k\le 3$ by definition, differentiate $\mathcal{V}$: trivial steps yield $\delta x/\delta\mathcal{V}=x/k\mathcal{V}$. Multiplying both sides by an arbitrary force F_x, the result is

$$\frac{F_x}{\delta\mathcal{V}/\delta x}=\frac{1}{k}\frac{F_x}{\mathcal{V}/x}\quad k=1,2,3.\tag{120}$$

Put then

$$\frac{F_x}{\mathcal{V}/x}\Rightarrow\frac{F_x}{A_{yz}}=P_x\quad\mathcal{V}=xA_{yz}\quad\text{whereas}\quad\frac{F_x\delta x}{\delta\mathcal{V}}=\frac{\epsilon}{\mathcal{V}}=\eta:\tag{121}$$

the first result defines the pressure exerted by F_x on the surface A_{yz}; the second result implies the work $F_x\delta x$ done by F_x to change the volume $\mathcal{V}$ by $\delta\mathcal{V}$ when x is stretched by δx. Note however that actually

$$\frac{F_x\delta x}{\delta\mathcal{V}}\Rightarrow\frac{F_x\delta x}{\delta\mathcal{V}}+\frac{F_y\delta y}{\delta\mathcal{V}}+\frac{F_z\delta z}{\delta\mathcal{V}}\quad\frac{F_xx}{\mathcal{V}}\Rightarrow\frac{F_xx}{\mathcal{V}}+\frac{F_yy}{\mathcal{V}}+\frac{F_zz}{\mathcal{V}};$$

as by symmetry the three addends are equivalent, the sums suggest a factor 3 multiplying both sides of (120) to obtain from (121) a result compliant with a true 3D effect. Is known the physical meaning of

$$P=\frac{k}{3}\eta.\tag{122}$$

2) Implement again the definition (118) and (14) to obtain for a charge of mass m traveling in the vacuum

$$a^2=\frac{\varepsilon^2c^4}{\theta e^4}=\left((pc)^2+(mc^2)^2\right)\frac{c^4}{\theta e^4}=\left(1+\frac{(p/m)^2}{c^2}\right)\frac{c^4(mc^2)^2}{\theta e^4}.$$

Since the last factor is of course acceleration, write the result having dimensions *velocity*/*time* as follows

$$a=\pm\frac{v'}{t+t_0}\sqrt{1+\frac{v^2}{c^2}}\quad\frac{v'}{t+t_0}=\frac{c^2(mc^2)}{\sqrt{\theta}e^2}$$

where t_0 is added to fufilll any possible boundary condition for *a*, e.g. at $t=0$, without divergence. So

$$v' = \frac{v_a}{\sqrt{1+\frac{v^2}{c^2}}} \quad v_a = (t+t_0)a \Rightarrow v' = \frac{v_a}{\sqrt{1+\frac{v_a^2}{c^2}}} \quad v^2 = v_a^2$$

reduces to the classical $v' = (t+t_0)a$ for $v_a \ll c$. Note that even if $m=0$, it is still possible to write

$$a = \frac{pc^3}{\theta e^2} = \frac{hc^3}{\theta\lambda e^2} = \frac{c^2/\lambda}{\theta\alpha} \tag{123}$$

whose physical meaning is under investigation; reasonably this condition concerns a field rather than a charged particle. Preliminary considerations suggest that the differential $\delta a = Y\delta(1/\lambda)$, being $Y = c^2/\theta\alpha$ a constant, yields owing to (7) $v_g\delta a/Y = \delta v$ and thus

$$\frac{v_g}{v}\frac{\delta a}{Y} = \frac{\delta v}{v} \Rightarrow \frac{\delta\varphi}{c^2} = -\frac{\delta v}{v} \quad -\delta\varphi = \frac{\theta\alpha v_g\delta a}{v},$$

which matches (18) and (19). In other words, c^2/λ has to do with the definition of gravitational potential governing the gravitational red shift.

3) Implement (9), the equation through which have been calculated the electron energy levels, to find now

$$const = \frac{mG}{nc^2}v\delta r \Rightarrow 0 = \delta\left(\frac{mG}{nc^2}v\delta r\right)$$

i.e.

$$0 = \frac{G\delta m}{nc^2}v\delta r - \frac{mG}{n^2c^2}v\delta r\delta n \quad \delta n = \pm integer;$$

clearly the sign of δn depends upon the chances of increasing or decreasing *n*. Write thus

$$\frac{G\delta m}{nc^2}v\delta r = \frac{mG}{\delta r}\frac{\delta r^2}{c^2}v = \frac{const}{n}\delta n \quad const = \frac{\hbar G}{c^2}:$$

multiplying both sides by an arbitrary mass m' one finds

$$\frac{m'\delta mG}{\delta r} = \frac{const}{n}\frac{m'c^2}{v\delta r^2}\delta n.$$

Therefore the result is

$$\frac{m'm''G}{\delta r} = \pm\frac{\mathcal{E}}{n}|\delta n| \quad \mathcal{E} = const\frac{m'c^2}{v\delta r^2} \quad m'' = \delta m,$$

because the difference $\delta m = m_1 - m_0$ of two masses is clearly a new mass itself. With the minus sign, the left hand side reports the Newton energy, which however is defined now via δr and not *r*. The main problem of the classical Newton law is not the fact that it is approximate, several equations of physics are acceptable even so; the main problem, which worried Newton himself, is that the deterministic *r* implies an instantaneous action at a distance. On the one hand, now the uncertainty range implies propagation time of an appropriate force vector (graviton?), as (1) require $\delta r = (\delta\varepsilon/\delta p_r)\delta t = v_r \delta t$ ($v_r = c$?). On the other hand the Newton energy appears to be quantized via $|\delta n|$, the difference of integers is an integer itself. The idea is that the number n of allowed quantum states significantly determines the gravity force, as in general $\delta \dot{p}_x = -n\hbar\delta\dot{x}/\delta x^2 + \hbar\delta\dot{n}/\delta x$.

6. Conclusion

The matter tells the space time how to deform, $\delta\dot{x}$, the space time tells the matter how to move, δp_x, and how to change its number of allowed quantum states, $\delta\dot{n}$ [6]. This is reasonable because (1) imply the equivalence principle as a corollary. Write indeed $\delta\dot{x} = \dot{x}_1 - \dot{x}_0$ and let for simplicity $\dot{x}_0 = 0$, *i.e.* the upper range boundary only is time dependent, which however is enough to give rise a force field F_x in δx due to $\delta\dot{p}_x \neq 0$; indeed even so $\delta\dot{x} \neq 0$. An observer sitting on x_1 experiences F_x and concludes that he moves with respect to the origin O of the arbitrary reference system R where is defined δx. Another observer sitting on x_0 also experiences the same force although he is at rest: so he concludes that he is in a gravity field. As of course F_x is the same for both, the conclusion is that gravity field is indistinguishable from accelerating system. This holds also for a local force when the size of $\delta x \to 0$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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