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Asymptotic and Oscillatory Analysis of Fourth-Order Nonlinear Differential Equations with p-Laplacian-like Operators and Neutral Delay Arguments

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Abstract: This paper delves into the asymptotic and oscillatory behavior of all classes of solutions of fourth-order nonlinear neutral delay differential equations in the noncanonical form with damping terms. This research aims to improve the relationships between the solutions of these equations and their corresponding functions and derivatives. By refining these relationships, we unveil new insights into the asymptotic properties governing these solutions. These insights lead to the establishment of improved conditions that ensure the nonexistence of any positive solutions to the studied equation, thus obtaining improved oscillation criteria. In light of the broader context, our findings extend and build upon the existing literature in the field of neutral differential equations. To emphasize the importance of the results and their applicability, this paper concludes with some examples.

Keywords: nonlinear differential equations; asymptotic and oscillatory analysis; fourth-order; p-Laplacian-like operators; neutral delay arguments

MSC: 34C10; 34K11

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1. Introduction

Neutral differential equations (NDDEs) are one of the most apparent subclasses of differential equations for application and modeling. These equations possess a distinct characteristic from ordinary differential equations, where the rate of change of the dependent variable at a given time depends not only on the current value but also on its past values, introducing a temporal delay argument into the equations. The delay arguments reflect real-world scenarios where systems exhibit memory effects, making NDDEs a critical tool in modeling and analyzing systems with such characteristics [1,2].

Oscillation theory is a significant branch of mathematics concerned with studying the behavior of oscillatory and nonoscillatory systems. Fundamentally, this theory studies systems that repeatedly and arbitrarily oscillate around a central value or between two or more states. This behavior is common in nature and technology, manifesting in mechanical, electrical, and biological systems. For example, in mechanical systems, oscillation theory appears in the stability of electrical circuits; in ecological models, it appears in the population dynamics of species [3,4].

In recent decades, studying the oscillation of solutions of different classifications of differential equations has seen significant development and growth. Among these different classifications, differential equations with delay arguments have received considerable interest from researchers, as shown in [5–7]. Likewise, the study of NDDEs has been a focal point in a range of publications [8–10]. Additionally, the research community has demonstrated notable evolution in the investigation of odd-order differential equations, exemplified biny [11–13]. In contrast, the oscillation behavior of even-order equations has been examined, as documented in [14–16]. Furthermore, the dynamics of damping equations have been explored, as indicated in [17,18].

In addition, the complexity of fourth-order differential equations often requires resorting to studying the behavior of the solutions, regardless of finding them in a closed form. So, in this study, we focused on studying the behavior of the positive and oscillatory solutions of the following nonlinear fourth-order neutral delay differential equations with damped term

$$\left(a(t)|z'''(t)|^{p-1}z'''(t)\right)' + \rho(t)|z'''(t)|^{p-1}z'''(t) + q(t)|x(\kappa(t))|^{\beta-1}x(\kappa(t)) = 0, \quad (1)$$

for all $t \geq t_0$, $p > 0$, $\beta > 0$, and the corresponding function $z(t)$ defined as

$$z(t) = x(t) + \mathrm{c}(t)x(h(t)).$$

We based our studies on the following assumptions:

(H_1) $a \in C^1([t_0,\infty),(0,\infty))$, $a'(t) \geq 0$, $\mathrm{c} \in C([t_0,\infty),\mathbb{R})$, and $0 \leq \mathrm{c} < 1$;
(H_2) $h \in C([t_0,\infty),\mathbb{R})$, $\kappa \in C^1([t_0,\infty),\mathbb{R})$, $h(t) \leq t$, $\kappa(t) \leq t$, $\kappa'(t) > 0$, and $\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \kappa(t) = \infty$;
(H_3) $\rho, q \in C([t_0,\infty),(0,\infty))$, $q(t)$ is not eventually zero on $[t^*,\infty)$ for $t^* \geq t_0$.

Let $t_x = \min\{h(t),\kappa(t)\}$. A function $x(t) \in C^4([t_x,\infty),\mathbb{R})$, $t_x \geqslant t_0$, is called a solution of (1) if it has the property $a(t)|z'''(t)|^{p-1}z'''(t) \in C^1[t_x,\infty)$ and satisfies (1) on $[t_x,\infty)$. We only consider the nontrivial solutions of (1), which ensure

$$\sup\{|x(t)| : t \geq t_x\} > 0.$$

A solution of (1) is said to be oscillatory if it has an arbitrarily large zero point on $[t_x,\infty)$. Otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

Even-order differential equations have received great attention from researchers in the field of studying oscillatory behavior and the asymptotic properties of positive solutions. Chatzarakis et al. [19], Li et al. [20], and Bohner et al. [21] were interested in second-order differential equations, whether in canonical or noncanonical cases, and improvements continued to include sharp oscillation criteria, better than which could not be achieved for delayed and neutral delayed equations by Essam et al. [22] and Jadlovska [23], respectively. As for fourth-order equations, Grace et al. were interested in studying and improving the oscillation criteria and the asymptotic properties of positive solutions, relying on different techniques and methods [6,24]. For more information, please see [25–32].

Below, we provide a simple summary of the most important previous works that have played a fundamental role in developing the theory of oscillation of even-order differential equations.

In 1989, Philos [33] studied the second-order differential equation

$$x''(t) + q(t)x(t) = 0,$$

where they improved on Yan's work [34,35], extending Kamenev's criterion and giving one of the most famous oscillation criteria, which have inspired many works to date. One of these works is the work of Wang in 2001 [36], who relied on the generalized Riccati substitution and the integral averaging technique to give some improved oscillation criteria and study the asymptotic properties of the second-order differential equation

$$\left(a(t)|x'(t)|^{p-1}x'(t)\right)' + q(t)|x(t)|^{p-1}x(t) = 0. \quad (2)$$

On the other hand, Baculikova and Dzurina [37] introduced delay neutral arguments to (2) and used another approach based on the comparison theorem for the studied equation

$$\left(a(t)[x(t)+\mathrm{c}(t)x(h(t))]^{(n-1)}\right)'+q(t)x(\kappa(t))=0 \tag{3}$$

with known oscillatory equations of the first order to conclude that (3) is oscillatory if the condition

$$\liminf_{t\to\infty}\int_{\kappa(t)}^{t}Q_i(s)\mathrm{d}s>\frac{h_0+\mathrm{c}_0}{h_0\mathrm{e}}$$

is met, where $Q(s)=\min\{q(t),q(h(t))\},\ i=2,4,6,...,n-1$. The results obtained during this study relax some of the conditions imposed on coefficients compared to those in previous works. In 2011, Zhang et al. [38] used the generalized Riccati substitution and the integral averaging technique to derive some improved oscillation criteria for a class of the even-order half-linear delay differential equations with damping term

$$\left(a(t)\left|x^{(n-1)}(t)\right|^{p-1}x^{(n-1)}(t)\right)'+\rho(t)\left(|x(t)|^{p-1}x(t)\right)^{(n-1)}+q(t)|x(\kappa(t))|x(\kappa(t))=0, \tag{4}$$

where they provided that for the nonpositive continuous partial derivative with respect to the second variable $H(t,t_0)$, there exists functions $h(t,t_0)\in C([t_0,\infty),\mathbb{R})$, $\delta(s),\varpi(s)\in C^1([t_0,\infty),\mathbb{R}^+)$, $\theta\in(0,1)$, $M\in(0,\infty)$, and

$$G(s)=\theta M\kappa^{n-2}(t)\kappa'(t).$$

Then, the above equation oscillates if

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\delta(s)\varpi(s)q(s)\mathrm{d}s=\infty,$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\varpi(s)r(s)|h(t,s)|^p}{(H(t,s)\delta(s)G(s))}H(t,s)\delta(s)\varpi(s)q(s)\mathrm{d}s=\infty.$$

After that, the same authors improved their previous results and obtained that (4) is oscillatory if

$$\liminf_{t\to\infty}\int_{\kappa(t)}^{t}\frac{q(s)}{a(\kappa(s))}\left(\kappa^{n-1}(s)\right)^{p-1}\exp\left(\int_{\kappa(t)}^{s}\frac{\rho(u)}{a(u)}\mathrm{d}u\right)\mathrm{d}s>\frac{((n-1)!)^{p-1}}{\mathrm{e}}.$$

In 2023, Barakah et al. [39] studied the no ncanonical case for (3), where they found a new relationship between the solution and its corresponding function and used these relations to find that (3) oscillates or tends to zero if

$$\liminf_{t\to\infty}\int_{\kappa(t)}^{t}q_{n-3}(s)\left(\int_{t_1}^{s}q(u)\mathrm{c}_1(\kappa(u);n)\mathrm{d}u\right)\mathrm{d}s>\frac{h_0+\mathrm{c}_0}{h_0\mathrm{e}},$$

where n states for any non-negative integer and

$$\mathrm{c}_1(t;n)=\sum_{i=0}^{n}\left(\prod_{k=0}^{2i}\mathrm{c}\left(h^{[k]}(t)\right)\right)\left[\frac{1}{\mathrm{c}\left(h^{[2i]}(t)\right)}-\frac{\phi_2\left(h^{[2i+1]}(t)\right)}{\phi_2\left(h^{[2i]}(t)\right)}\right].$$

The primary aim of this study was to extend the scope of research on the oscillation of solutions of fourth-order differential equations to ensure the presence of damping terms and to obtain oscillation criteria that improve upon their predecessors when applied to mathematical models. The idea in our research was based on improving the relationships between positive solutions and their corresponding function and derivatives and using these relationships to derive criteria that guarantee the absence of any positive solution to (1), whether through the comparison theorems or the Riccati technique. After obtaining these criteria, we easily applied them and derived improved oscillation criteria that require fewer restrictions compared to the previous ones.

The inequalities presented herein are functions of the variable t, and we assume that these inequalities remain valid for sufficiently large values of t unless otherwise specified.

2. Preliminary Notations and Lemmas

In this section, we define some of the auxiliary functions and lemmas that we call while proving our main results.

Relying on the property of symmetry between positive and negative solutions, we only consider the positive solution for (1) when proving our results. So, let us define:

$$F^{[j]}(t) = F\left(F^{[j-1]}(t)\right)$$

for $j = 1, 2, ..., n$ and $F^{[0]}(t) := F(t)$, and

$$R(t) = a(t)E(t),$$

where

$$E(t) = \exp\left(\int_{t_0}^{t} \frac{\rho(\varrho)}{a(\varrho)} \mathrm{d}\varrho\right).$$

Furthermore, we define the following iterative function

$$\phi_0(t) := \int_t^{\infty} R^{-1/p}(\varrho)\mathrm{d}\varrho$$

$$\phi_i(t) := \int_t^{l} \phi_{i-1}(\varrho)\mathrm{d}\varrho, \qquad i = 1, 2,$$

$$\mathrm{c}_1(t; n) = \sum_{i=0}^{n} \left(\prod_{k=0}^{2i} \mathrm{c}\left(h^{[k]}(t)\right)\right) \left[\frac{1}{\mathrm{c}\left(h^{[2i]}(t)\right)} - \frac{\phi_2\left(h^{[2i+1]}(t)\right)}{\phi_2\left(h^{[2i]}(t)\right)}\right],$$

$$\mathrm{c}_2(t; n) := \sum_{i=0}^{n} \left(\prod_{k=0}^{2i} \mathrm{c}\left(h^{[k]}(t)\right)\right) \left[\frac{1}{\mathrm{c}\left(h^{[2i]}(t)\right)} - 1\right] \left(\frac{h^{[2i]}(t)}{t}\right)^{2/\epsilon_0},$$

$$Q(t) := E(t)q(t)\mathrm{c}_1^{\beta}(\kappa(t); n),$$

$$Q_1(t) := E(t)q(t)(1 - \mathrm{c}(\kappa(t)))^{\beta},$$

and

$$Q_2(t) := E(t)q(t)\mathrm{c}_2^{\beta}(\kappa(t); n).$$

During our work, we studied (1) in the noncanonical form, that is, in the case that

$$\phi_0(t) < \infty. \tag{5}$$

Lemma 1 ([40]). *Suppose that $x \in C^n([t_0, \infty), \mathbb{R}^+)$, $x^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0, \infty)$; then, there exists $t_1 \geq t_0$ such that $x^{(n-1)}(t)x^{(n)}(t) \leq 0$ for all $t_1 \geq t_0$. If $\lim_{t\to\infty} x(t) \neq 0$, then, for every $\delta \in (0, 1)$, there exists $t_\epsilon \in [t_1, \infty)$ such that*

$$x(t) \geq \frac{\epsilon}{(n-1)!} t^{n-1} \left| x^{(n-1)}(t) \right|,$$

for $t \in [t_\epsilon, \infty)$.

Lemma 2 ([41])**.** *Let the function $\hbar$ satisfy $\hbar^{(i)}(t) > 0$, $i = 0, 1, ..., n$ and $\hbar^{(n+1)}(t) \leq 0$ eventually. Then, for every $l \in (0, 1)$*

$$\frac{\hbar(t)}{\hbar'(t)} \geq \frac{lt}{2}$$

eventually.

Lemma 3 ([42])**.** *Let p be a ratio of two odd positive integers; $A > 0$ and B are constants. Then*

$$Bu - Au^{(p+1)/p} \leq \frac{p^p}{(p+1)^{p+1}} \frac{B^{p+1}}{A^p}, \; A > 0. \tag{6}$$

Lemma 4 ([40])**.** *Assume that x is an eventually positive solution of (1); then x satisfies eventually the following cases:*

$$\begin{aligned} &\mathrm{C}_1 \; : \; z^{(i)}(t) > 0 \text{ for } i = 0, 1, 3, \text{ and } z^{(4)}(t) < 0; \\ &\mathrm{C}_2 \; : \; z^{(i)}(t) > 0 \text{ for } i = 0, 1, 2, \text{ and } z'''(t) < 0; \\ &\mathrm{C}_3 \; : \; (-1)^i z^{(i)}(t) > 0 \text{ for } i = 0, 1, 2, 3; \end{aligned}$$

for $t \geqslant t_1 \geqslant t_0$.

Lemma 5 (see [43], Lemma 1)**.** *Assume that x is an eventually positive solution of (1). Then,*

$$x(t) > \sum_{i=0}^{n} \left(\prod_{k=0}^{2i} \mathrm{c}\left(h^{[k]}(t)\right) \right) \left[\frac{z\left(h^{[2i]}(\varrho)\right)}{\mathrm{c}\left(h^{[2i]}(\varrho)\right)} - z\left(h^{[2i+1]}(\varrho)\right) \right], \tag{7}$$

holds eventually.

Notation 1. The notation Ω_i (Category Ω_i) denotes the set comprising all solutions that eventually become positive and satisfy condition (C_i) for $i = 1, 2, 3$.

3. Nonexistence of Positive Solutions

In this section, we present some improved relationships between the solution and its corresponding function and derivatives, and we then use these improved relationships and the Riccati technique and comparison theorems to provide criteria that guarantee the absence of positive solutions for all the aforementioned cases (C_1, C_2, C_3). We consistently assume the validity of all functional inequalities for sufficiently large values. Furthermore, it is worth emphasizing that our focus on the ultimate positive solutions of Equation (1) suffices to ensure the integrity and soundness of our proofs.

3.1. Nonexistence of C_1*-Type Solutions*

In this subsection, we present some lemmas relevant to the asymptotic behavior of the positive solutions belonging to class (C_1).

Lemma 6. *Let $x \in \Omega_1$, and (5) holds. Then,*

$$\left(R(t)(z'''(t))^p\right)' + Q_1(t) z^\beta(\kappa(t)) \leq 0. \tag{8}$$

Proof. Let $x \in \Omega_1$; then, there exists $t_1 \geq t_0$ such that $x(h(t)) > 0$ and $x(\kappa(t)) > 0$ for $t \geq t_1$. Multiplying both sides of (1) by $E(t)$, we have the following equation without a damped term:

$$\left(R(t)|z'''(t)|^{p-1} z'''(t)\right)' = -E(t) q(t) x^\beta(\kappa(t)), \quad t \geq t_0.$$

Since $z'''(t) > 0$, this inequality simplifies to

$$\left(R(t)\left(z'''(t)\right)^p\right)' \leq -E(t)q(t)x^\beta(\kappa(t)), \tag{9}$$

which implies that $R(t)(z'''(t))^p$ is nonincreasing function. By using the definition of z, we can deduce that

$$x(t) = z(t) - \mathrm{c}(t)x(h(t)) \geq z(t) - \mathrm{c}(t)z(h(t)).$$

Since $z'(t) > 0$, and $h(t) \leq t$, we can infer that $z(t) \geq z(h(t))$. Combining this with the previous inequality, we have

$$x(t) \geq (1 - \mathrm{c}(t))z(t)$$

and

$$x(\kappa(t)) \geq (1 - \mathrm{c}(\kappa(t)))z(\kappa(t)). \tag{10}$$

Using (10) with (9), we have

$$\begin{aligned}\left(R(t)\left(z'''(t)\right)^p\right)' &= -E(t)q(t)x^\beta(\kappa(t)) \\ &\leq -E(t)q(t)(1 - \mathrm{c}(\kappa(t)))^\beta z^\beta(\kappa(t)) \\ &= -Q_1(t)z^\beta(\kappa(t)),\end{aligned}$$

and this completes the proof. □

Lemma 7. *Assume that $p \geq 1$, and the differential equation*

$$\omega'(t) + \left(\frac{\lambda_0\kappa^3(t)}{6R^{1/p}(t)}\right)^\beta Q_1(t)\omega^{\beta/p}(t) = 0 \tag{11}$$

is oscillatory for some $\lambda_0 \in (0,1)$. Then, $\Omega_1 = \varnothing$.

Proof. Suppose the contrary that $x \in \Omega_1$. In other words, there exists a $t_1 \geq t_0$ such that $x(h(t)) > 0$ and $x(\kappa(t))$ for all $t \geq t_1$. Using Lemma 1, we have

$$z(t) \geq \frac{\lambda_0}{6}t^3 z'''(t), \tag{12}$$

for all $\lambda_0 \in (0,1)$ and sufficiently large t. Using (12) in (8), we find that

$$\left(R(t)\left(z'''(t)\right)^p\right)' + \left(\frac{\lambda_0\kappa^3(t)}{6R^{1/p}(t)}\right)^\beta Q_1(t)\left(R^{1/p}(t)z'''(\kappa(t))\right)^\beta \leq 0.$$

Let $\omega(t) = R(t)(z'''(t))^p$ in the last inequality. We obtain that $\omega(t)$ is a positive solution for the delay differential inequality

$$\omega'(t) + \left(\frac{\lambda_0\kappa^3(t)}{6R^{1/p}(t)}\right)^\beta Q_1(t)\omega^{\beta/p}(\kappa(t)) \leq 0.$$

However, according to Corollary 1 in [44], there exists a positive solution of (11), a contradiction. And this completes the proof. □

Lemma 8. *Assume that (5) holds. If there exists a positive nondecreasing function $\xi \in C^1([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left(\xi(\varrho)Q_1(\varrho) - \frac{2^v}{(v+1)^{v+1}}\frac{R(\theta(\varrho))(\xi'(\varrho))^{v+1}}{(K\lambda_1\xi(\varrho)\kappa'(\varrho)\kappa^2(\varrho))^v}\right)\mathrm{d}\varrho = \infty, \tag{13}$$

then, $\Omega_1 = \varnothing$.

Proof. Suppose on the contrary that $x \in \Omega_1$. In other words, there exists a $t_1 \geq t_0$ such that $x(h(t)) > 0$ and $x(\kappa(t))$ for all $t \geq t_1$. Define a function $w(t)$ by

$$w(t) := \xi(t)\frac{R(t)(z'''(t))^p}{z^\beta(\kappa(t))},\ t \geq t_1. \tag{14}$$

Then, $w(t) > 0$ and

$$\begin{aligned} w'(t) &= \xi'(t)\frac{R(t)(z'''(t))^p}{z^\beta(\kappa(t))} + \xi(t)\frac{\left(R(t)(z'''(t))^p\right)'}{z^\beta(\kappa(t))} \\ &\quad -\beta\kappa'(t)\xi(t)\frac{R(t)(z'''(t))^p}{z^{\beta+1}(\kappa(t))}z'(\kappa(t)). \end{aligned} \tag{15}$$

By using Lemma 1, we obtain

$$z'(t) \geq \frac{\lambda_1}{2}t^2 z'''(t),$$

or, equivalently,

$$z'(\kappa(t)) \geq \frac{\lambda_1}{2}\kappa^2(t)z'''(\kappa(t)). \tag{16}$$

By using (8), (14), and (16) in (15), we have

$$\begin{aligned} w'(t) &\leq -\xi(t)Q_1(t) + \frac{\xi'(t)}{\xi(t)}w(t) \\ &\quad -\frac{\lambda_1}{2}\beta\kappa'(t)\kappa^2(t)\xi(t)\frac{R(t)(z'''(t))^p}{z^{\beta+1}(\kappa(t))}z'''(\kappa(t)). \end{aligned} \tag{17}$$

For this inequality, we first consider the case $p < \beta$. But, $R(t)(z'''(t))^p$ is a positive nonincreasing function, so

$$R^{1/p}(t)z'''(t) \leq R^{1/p}(\kappa(t))z'''(\kappa(t)).$$

In view of (17), we have

$$\begin{aligned} w'(t) &\leq -\xi(t)Q_1(t) + \frac{\xi'(t)}{\xi(t)}w(t) \\ &\quad -\frac{\lambda_1}{2}\beta\frac{\kappa'(t)\kappa^2(t)z^{(\beta-p)/p}(\kappa(t))}{(\xi(t)R(\kappa(t)))^{1/p}}w^{(p+1)/p}(t). \end{aligned}$$

Since $z(\kappa(t))$ is an increasing function, there exist constants $K_1 > 0$ and $t_2 \geq t_1$ such that

$$z^{(\beta-p)/p}(\kappa(t)) \geq K_1,\ t \geq t_2. \tag{18}$$

Hence, we obtain

$$w'(t) \leq -\xi(t)Q_1(t) + \frac{\xi'(t)}{\xi(t)}w(t) - \frac{\lambda_1 p}{2}K_1\frac{\kappa'(t)\kappa^2(t)}{(\xi(t)R(\kappa(t)))^{1/p}}w^{(p+1)/p}(t). \tag{19}$$

If $p = \beta$, then $K_1 = 1$; thus, (19) still holds.

Now, if $p > \beta$, $r'(t) \geq 0$ implies that $R'(t) \geq 0$. Recall that $\left(R(t)(z'''(t))^p\right)' \leq 0$, hence $y^{(4)}(t) \leq 0$, which implies that

$$\left(z'''(t)\right)^{(\beta-p)/\beta} \geq K_2,\ t \geq t_3. \tag{20}$$

By combining (17) and (20), we then have

$$\begin{aligned} w'(t) &\leq -\xi(t)Q_1(t) + \frac{\xi'(t)}{\xi(t)}w(t) - \frac{\lambda_1\beta}{2}\frac{\kappa'(t)\kappa^2(t)}{(\xi(t)R(t))^{1/\beta}}\left(z'''(\kappa(t))\right)^{(\beta-p)/\beta}w^{(\beta+1)/\beta}(t) \\ &\leq -\xi(t)Q_1(t) + \frac{\xi'(t)}{\xi(t)}w(t) - \frac{\lambda_1\beta}{2}K_2\frac{\kappa'(t)\kappa^2(t)}{(\xi(t)R(t))^{1/\beta}}w^{(\beta+1)/\beta}(t), \end{aligned}$$

which, tohaveher with (19), implies that

$$w'(t) \leq -\xi(t)Q_1(t) + \frac{\xi'(t)}{\xi(t)}w(t) - \frac{\lambda_1\nu}{2}K\frac{\kappa'(t)\kappa^2(t)}{(\xi(t)R(\theta(t)))^{1/\nu}}w^{(\nu+1)/\nu}(t),\ t \geq t_3, \tag{21}$$

where $\nu = \min\{p,\beta\}$, $K = \min\{K_1, K_2\}$, and

$$\theta(t) = \begin{cases} t, & p > \beta, \\ \kappa(t), & p \leq \beta. \end{cases}$$

Using Lemma 3 where $B = \xi'(t)/\xi(t)$, $A = \lambda_1\nu K\kappa'(t)\kappa^2(t)/2(\xi(t)R(\theta(t)))^{1/v}$, and $u = w(t)$, we have

$$\frac{\xi'(t)}{\xi(t)}w(t) - \frac{\lambda_0\nu}{2}K\frac{\kappa'(t)\kappa^2(t)}{(\xi(t)R(\theta(t)))^{1/v}}w^{(v+1)/v}(t) \leq \frac{2^\nu}{(v+1)^{v+1}}\frac{R(\theta(t))(\xi'(t))^{v+1}}{(K\lambda_1\xi(t)\kappa'(t)\kappa^2(t))^{v}},$$

which, with (21), gives

$$w'(t) \leq -\xi(t)Q_1(t) + \frac{2^v}{(v+1)^{v+1}}\frac{R(\theta(t))(\xi'(t))^{v+1}}{(K\lambda_1\xi(t)\kappa'(t)\kappa^2(t))^{v}}.$$

Integrating this inequality from t_3 to t, we obtain

$$w(t) \leq w(t_3) - \int_{t_3}^{t}\left(\xi(\varrho)Q_1(\varrho) - \frac{2^v}{(v+1)^{v+1}}\frac{R(\theta(\varrho))(\xi'(\varrho))^{v+1}}{(K\lambda_1\xi(\varrho)\kappa'(\varrho)\kappa^2(\varrho))^{v}}\right)\mathrm{d}\varrho.$$

By taking $t \to \infty$ in the above inequality, we then obtain a contradiction with (13). The proof is complete. □

Lemma 9. *Assume that (5) hold. If there exists a positive nondecreasing function $\xi \in C^1([t_0,\infty), (0,\infty))$, such that*

$$\limsup_{t\to\infty}\int_{t_3}^{t} Q_1(t)\mathrm{d}\varrho = \infty, \tag{22}$$

then $\Omega_1 = \varnothing$.

Proof. Condition (22) follows by substituting $\xi(t) = 1$ into (13). And this completes the proof. □

3.2. Nonexistence of C_2-Type Solutions

In this subsection, we present some lemmas relevant to the asymptotic behavior of the positive solutions belonging to the class (C_2).

Lemma 10. *Let $x \in \Omega_2$, and (5) holds. Then, eventually,*
($N_{4,1}$) *$z(t) \geq \epsilon_0 tz'(t)$;*
($N_{4,2}$) *$z''(t) \geq -R^{1/p}(t)\phi_0(t)z'''(t)$;*
($N_{4,3}$) *$z''(t)/\phi_0(t)$ is increasing.*

Proof. Assume that $x \in \Omega_2$.
$(\mathrm{N}_{4,1})$ Using Lemma 2 with $n = 2$ and $x = z$, we have

$$z(t) \geq \epsilon_0 t z'(t).$$

$(\mathrm{N}_{4,2})$ Since $R^{1/p}(t)z'''(t)$ is a decreasing function, we have

$$z''(t) \geq -\int_t^\infty z'''(\varrho)\mathrm{d}\varrho \geq -R^{1/p}(t)\phi_0(t)z'''(t).$$

$(\mathrm{N}_{4,3})$ From $(\mathrm{N}_{4,2})$, we obtain

$$\left(\frac{z''(t)}{\phi_0(t)}\right)' = \frac{R^{1/p}(t)\phi_0(\varrho)z'''(t) + z''(t)}{R^{1/p}(t)\phi_0^2(\varrho)} \geq 0,$$

The proof is complete. □

Lemma 11. *Let $x \in \Omega_2$, and (5) holds. Then, eventually,*
$(\mathrm{N}_{5,1})$ $x(t) \geq \mathrm{c}_2(t;n)z(t)$;
$(\mathrm{N}_{5,2})$ $\left(R(\varrho)(-z'''(t))^p\right)' - Q_2(t)z^p(\kappa(\varrho)) \geq 0$.

Proof. Assume that $x \in \Omega_2$.
$(\mathrm{N}_{5,1})$ From Lemma 5 , we find that (7) holds. Based on the properties of solutions in the class Ω_2, we conclude that $z\left(h^{[2i]}(t)\right) \geq z\left(h^{[2i+1]}(t)\right)$ for $i = 1, 2, ..., n$. Thus, (7) becomes

$$x(t) > \sum_{i=0}^{n}\left(\prod_{k=0}^{2i}\mathrm{c}\left(h^{[k]}(t)\right)\right)\left[\frac{1}{\mathrm{c}(h^{[2i]}(t))} - 1\right]z\left(h^{[2i]}(t)\right). \tag{23}$$

Using $(\mathrm{N}_{4,1})$, we obtain

$$z\left(h^{[2i]}(t)\right) \geq \left(\frac{h^{[2i]}(t)}{t}\right)^{2/\epsilon_0} z(t),$$

with (23), this gives

$$\begin{aligned} x(t) &> \sum_{i=0}^{n}\left(\prod_{k=0}^{2i}\mathrm{c}\left(h^{[k]}(t)\right)\right)\left[\frac{1}{\mathrm{c}(h^{[2i]}(t))} - 1\right]\left(\frac{h^{[2i]}(t)}{t}\right)^{2/\epsilon_0} z(t) \\ &= \mathrm{c}_2(t;n)z(t). \end{aligned}$$

$(\mathrm{N}_{5,2})$ Since $z'''(t) < 0$, from (41), we obtain

$$\left(R(t)(-z'''(t))^p\right)' = E(t)q(t)x^\beta(\kappa(t)) \geq 0. \tag{24}$$

By using $(\mathrm{N}_{5,1})$, we can deduce

$$\begin{aligned} \left(R(t)(-z'''(t))^p\right)' &= E(t)q(t)x^\beta(\kappa(t)) \\ &\geq E(t)q(t)\mathrm{c}_2^\beta(\kappa(t);n)z^\beta(\kappa(t)) \\ &= Q_2(t)z^\beta(\kappa(t)), \end{aligned}$$

The proof is complete. □

Lemma 12. *Assume that $p \geq 1$. If there exists a positive function $\widetilde{\xi}(t) \in C^1([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\Psi(\varrho) - \frac{R(\varrho)\widetilde{\xi}(\varrho)}{(p+1)^{p+1}} \left(\frac{\widetilde{\xi}'(\varrho)}{\widetilde{\xi}(\varrho)} + \frac{(1+p)}{R^{1/p}(\varrho)\phi_0(\varrho)} \right)^{p+1} \right) \mathrm{d}\varrho = \infty, \tag{25}$$

holds for some $\lambda_2 \in (0,1)$ and any positive constants M_1 and M_2, where

$$\Psi(t) := \widetilde{\xi}(t) Q_2(t) \zeta(t) \left(\frac{\lambda_2}{2} \kappa^2(t) \right)^{\beta} - \frac{(p-1)\widetilde{\xi}(t)}{R^{1/p}(t)\phi_0^{p+1}(t)},$$

then $\Omega_2 = \varnothing$.

Proof. Suppose on the contrary that $x \in \Omega_2$. In other words, there exists a $t_1 \geq t_0$ such that $x(h(t)) > 0$ and $x(\kappa(t))$ for all $t \geq t_1$. Since $z'''(t) < 0$, (41) becomes

$$\left(R(t)\left(-z'''(t)\right)^{p} \right)' = E(t) q(t) x^{p}(\kappa(t)) \geq 0.$$

From (10), we deduce that

$$\left(R(t)\left(-z'''(t)\right)^{p} \right)' \geq Q_2(t) z^{p}(\kappa(t)). \tag{26}$$

Since $R(t)(-z'''(t))^p$ is increasing, $R(t)(z'''(t))^p$ is decreasing. Therefore

$$\begin{aligned} z''(l) - z''(t) &= \int_t^l \frac{1}{R^{1/p}(\varrho)} \left(R(\varrho)\left(z'''(\varrho)\right)^{p} \right)^{1/p} \mathrm{d}\varrho \\ &\leq R^{1/p}(t) z'''(t) \int_t^l \frac{1}{R^{1/p}(\varrho)} \mathrm{d}\varrho, \end{aligned}$$

by taking $l \to \infty$, we have

$$-z''(t) \leq R^{1/p}(t) z'''(t) \phi_0(t). \tag{27}$$

Now, let us define the function $G(t)$ as

$$G(t) := \widetilde{\xi}(t) \left(-\frac{R(t)(-z'''(t))^{p}}{(z''(t))^{p}} + \frac{1}{\phi_0^{p}(t)} \right). \tag{28}$$

From (28), we have $G(t) > 0$, for $t \geq t_1$. Therefore, we have

$$\begin{aligned} G'(t) \leq\ & \frac{\widetilde{\xi}'(t)}{\widetilde{\xi}(t)} G(t) - \widetilde{\xi}(t) Q_2(t) \frac{z^{\beta}(\kappa(t))}{(z''(t))^{p}} \\ & - p \frac{\widetilde{\xi}(t) R(t)(-z'''(t))^{p+1}}{(z''(t))^{p+1}} + \frac{p\widetilde{\xi}(t)}{\phi_0^{p+1}(t) R^{1/p}(t)}. \end{aligned}$$

Using (28), we deduce that

$$\begin{aligned} G'(t) &\leq \frac{\widetilde{\xi}'(t)}{\widetilde{\xi}(t)}G(t) - \widetilde{\xi}(t)Q_2(t)\frac{z^{\beta}(\kappa(t))}{(z''(t))^{p}} \\ &\quad -p\frac{\widetilde{\xi}(t)}{R^{1/p}(t)}\left(\frac{G(t)}{\widetilde{\xi}(t)} - \frac{1}{\phi_0^p(t)}\right)^{(p+1)/p} + \frac{p\widetilde{\xi}(t)}{\phi_0^{p+1}(t)R^{1/p}(t)} \\ &= \frac{\widetilde{\xi}'(t)}{\widetilde{\xi}(t)}G(t) - \widetilde{\xi}(t)Q_2(t)\frac{z^{\beta}(\kappa(t))}{(z''(\kappa(t)))^{\beta}}\frac{(z''(\kappa(t)))^{p}}{(z''(t))^{p}}(z''(\kappa(t)))^{\beta-p} \\ &\quad -p\frac{\widetilde{\xi}(t)}{R^{1/p}(t)}\left(\frac{G(t)}{\widetilde{\xi}(t)} - \frac{1}{\phi_0^p(t)}\right)^{(p+1)/p} + \frac{p\widetilde{\xi}(t)}{\phi_0^{p+1}(t)R^{1/p}(t)}. \end{aligned} \tag{29}$$

From Lemma 1, we obtain

$$z(t) \geq \frac{\lambda_2}{2}t^2 z''(t), \tag{30}$$

i.e.,

$$\frac{z''(\kappa(t))}{z''(t)} \geq 1. \tag{31}$$

By using (30) and (31) in (29), it becomes clear that

$$\begin{aligned} G'(t) &\leq \frac{\widetilde{\xi}'(t)}{\widetilde{\xi}(t)}G(t) - \widetilde{\xi}(t)Q_2(t)\left(\frac{\lambda_2}{2}\kappa^2(t)\right)^{\beta}(z''(\kappa(t)))^{\beta-p} \\ &\quad -\frac{p\widetilde{\xi}(t)}{R^{1/p}(t)}\left(\frac{G(t)}{\widetilde{\xi}(t)} - \frac{1}{\phi_0^p(t)}\right)^{(p+1)/p} + \frac{p\widetilde{\xi}(t)}{\phi_0^{p+1}(t)R^{1/p}(t)}. \end{aligned} \tag{32}$$

In the case that $p < \beta$, by using the increasing property of $R(t)(-z'''(t))^p$ for $t \geq t_1$, we obtain

$$R(t)(-z'''(t))^p \geq R(t_1)(-z'''(t_1))^p = M_1.$$

That is

$$R^{1/p}(t)z'''(t) \leq R^{1/p}(t_1)z'''(t_1) = -M_1^{1/p} < 0,$$

then

$$R^{1/p}(t)z'''(t) \leq -M_1^{1/p}.$$

If we divide this inequality by $R^{1/p}(t)$ and integrate the resulting inequality from t to l, we obtain

$$z''(u) \leq z''(t) - M_2^{1/p}\int_t^u \frac{1}{R^{1/p}(t)}\mathrm{d}\varrho.$$

Letting $u \to \infty$ and using (5), we have

$$0 \leq z''(t) - M_1^{1/p}\phi_0(t),$$

which yields

$$z''(t) \geq M_1^{1/p}\phi_0(t).$$

Thus, we conclude that

$$(z''(t))^{\beta-p} \geq M_1^{(\beta-p)/p}\phi_0^{\beta-p}(t). \tag{33}$$

Using (33) in (32), we obtain

$$G'(t) \le \frac{\widetilde{\xi}'(t)}{\widetilde{\xi}(t)}G(t) - \widetilde{\xi}(t)Q_2(t)\left(\frac{\lambda_2}{2}\kappa^2(t)\right)^{\beta} M_1^{(\beta-p)/p}\phi_0^{\beta-p}(t) - \frac{p\widetilde{\xi}(t)}{R^{1/p}(t)}\left(\frac{G(t)}{\widetilde{\xi}(t)} - \frac{1}{\phi_0^p(t)}\right)^{(p+1)/p} + \frac{p\widetilde{\xi}(t)}{\phi_0^{p+1}(t)R^{1/p}(t)}. \tag{34}$$

In the case that $p=\beta$, it is easy to see that $(z''(t))^{\beta-p}=1$, thus, (34) still holds.

In the case that $p>\beta$, since x is a nonincreasing positive function, there exists a $M_2>0$ such that $z''(t)\le M_2$, which implies that

$$(z''(t))^{\beta-p} \ge M_2^{\beta-p}. \tag{35}$$

By using (35) in (32), we have

$$G'(t) \le \frac{\widetilde{\xi}'(t)}{\widetilde{\xi}(t)}G(t) - \widetilde{\xi}(t)Q_2(t)\left(\frac{\lambda_2}{2}\kappa^2(t)\right)^{\beta} M_2^{\beta-p} - \frac{p\widetilde{\xi}(t)}{R^{1/p}(t)}\left(\frac{G(t)}{\widetilde{\xi}(t)} - \frac{1}{\phi_0^p(t)}\right)^{(p+1)/p} + \frac{p\widetilde{\xi}(t)}{\phi_0^{p+1}(t)R^{1/p}(t)}, \tag{36}$$

which, tohaveher with (34), implies that

$$G'(t) \le \frac{\widetilde{\xi}'(t)}{\widetilde{\xi}(t)}G(t) - \widetilde{\xi}(t)Q_2(t)\left(\frac{\lambda_2}{2}\kappa^2(t)\right)^{\beta} \zeta(t) - \frac{p\widetilde{\xi}(t)}{R^{1/p}(t)}\left(\frac{G(t)}{\widetilde{\xi}(t)} - \frac{1}{\phi_0^p(t)}\right)^{(p+1)/p} + \frac{p\widetilde{\xi}(t)}{\phi_0^{p+1}(t)R^{1/p}(t)}, \tag{37}$$

where

$$\zeta(t) = \begin{cases} 1 & \text{if } p=\beta, \\ c_1 & \text{if } p>\beta, \\ c_2\phi_0^{\beta-p}(t) & \text{if } p<\beta. \end{cases}$$

By using the inequality

$$A^{(p+1)/p} - (A-B)^{(p+1)/p} \le \frac{B^{1/p}}{p}[(1+p)A-B],\ AB>0,$$

with $A=G(t)/\widetilde{\xi}(t)$ and $B=1/\phi_0^p(t)$, we have

$$G'(t) \le \frac{\widetilde{\xi}'(t)}{\widetilde{\xi}(t)}G(t) - \widetilde{\xi}(t)Q_2(t)\left(\frac{\lambda_2}{2}\kappa^2(t)\right)^{\beta} \zeta(t) + \frac{p\widetilde{\xi}(t)}{\phi_0^{p+1}(t)R^{1/p}(t)} - \frac{p\widetilde{\xi}(t)}{R^{1/p}(t)}\left(\left(\frac{G(t)}{\widetilde{\xi}(t)}\right)^{(p+1)/p} - \frac{1}{\phi_0(t)p}\left[(1+p)\frac{G(t)}{\widetilde{\xi}(t)} - \frac{1}{\phi_0^p(t)}\right]\right),$$

which is

$$G'(t) \le \left(\frac{\widetilde{\xi}'(t)}{\widetilde{\xi}(t)} + \frac{(1+p)}{R^{1/p}(t)\phi_0(t)}\right)G(t) - \widetilde{\xi}(t)Q_2(t)\left(\frac{\lambda_2}{2}\kappa^2(t)\right)^{\beta}\zeta(t) - \frac{pG^{(p+1)/p}(t)}{R^{1/p}(t)\widetilde{\xi}^{1/p}(t)} - \frac{\widetilde{\xi}(t)}{R^{1/p}(t)\phi_0^{p+1}(t)} + \frac{p\widetilde{\xi}(t)}{R^{1/p}(t)\phi_0^{p+1}(t)}. \tag{38}$$

Using Lemma 3 where $B = \xi'(t)/\xi(t) + (1+p)/R^{1/p}(t)\phi_0(t)$, $A = p/R^{1/p}(t)\widetilde{\xi}^{1/p}(t)$ and $u = G(t)$, we have

$$\begin{aligned} G'(t) &\leq -\widetilde{\xi}(t)Q_2(t)\zeta(t)\left(\frac{\lambda_2}{2}\kappa^2(t)\right)^{\beta} + \frac{(p-1)\widetilde{\xi}(t)}{R^{1/p}(t)\phi_0^{p+1}(t)} \\ &\quad + \frac{R(t)\widetilde{\xi}(t)}{(p+1)^{p+1}}\left(\frac{\widetilde{\xi}'(t)}{\widetilde{\xi}(t)} + \frac{(1+p)}{R^{1/p}(t)\phi_0(t)}\right)^{p+1}. \end{aligned} \tag{39}$$

Integrating (39) from t_1 to t, we have

$$\int_{t_1}^{t}\left(\Psi(\varrho) - \frac{R(\varrho)\widetilde{\xi}(\varrho)}{(p+1)^{p+1}}\left(\frac{\widetilde{\xi}'(\varrho)}{\widetilde{\xi}(\varrho)} + \frac{(1+p)}{R^{1/p}(\varrho)\phi_0(\varrho)}\right)^{p+1}\right)\mathrm{d}\varrho \leq G(t_1),$$

which contradicts (25). This completes the proof. □

Lemma 13. *Assume that $p \geq 1$. If*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(Q_2(\varrho)\zeta(\varrho)\left(\frac{\lambda_2}{2}\kappa^2(\varrho)\right)^{\beta} - \frac{p}{R^{1/p}(\varrho)\phi_0^{p+1}(\varrho)}\right)\mathrm{d}\varrho = \infty, \tag{40}$$

holds for some $\lambda_2 \in (0,1)$ and any positive constants M_1 and M_2; then, $\Omega_2 = \varnothing$.

Proof. Condition (40) follows by substituting $\widetilde{\xi}(t) = 1$ into (25). And this completes the proof. □

3.3. Nonexistence of C_3*-Type Solutions*

In this subsection, we present some lemmas relevant to the asymptotic behavior of the positive solutions belonging to the class (C_3).

Lemma 14. *Let $x \in \Omega_3$ and (5) holds. Then,*
$(N_{1,1})$ *$z(t)/\phi_2(t)$ is increasing;*
$(N_{1,2})$ *$(-1)^{i+1}z^{(2-i)}(t) \leq R^{1/p}(t)z'''(t)\phi_i(t)$, for $i = 0,1,2$.*

Proof. Let $x \in \Omega_3$, then there exists a $t_1 \geq t_0$ such that $x(h(t)) > 0$ and $x(\kappa(t)) > 0$ for $t \geq t_1$. Multiplying both sides of (1) by $E(t)$, we have the following equation without a damped term:

$$\left(R(t)|z'''(t)|^{p-1}z'''(t)\right)' + E(t)q(t)x^{\beta}(\kappa(t)) = 0, \quad t \geq t_0. \tag{41}$$

But, as $z'''(t) < 0$, then

$$\left(R(t)(-z'''(t))^{p}\right)' \geq E(t)q(t)x^{\beta}(\kappa(t)). \tag{42}$$

$(N_{1,1})$ It follows from (42) that

$$R^{1/p}(\varrho)z'''(\varrho) \leq R^{1/p}(t)z'''(t), \quad \varrho \geq t \geq t_1$$

Dividing the above inequality by $R^{1/p}(\varrho)$, we obtain

$$z'''(\varrho) \leq \frac{R^{1/p}(t)z'''(t)}{R^{1/p}(\varrho)}.$$

Integrating the above inequality from t to ∞, we have

$$-z''(t) \leq R^{1/p}(t)z'''(t)\int_t^\infty R^{-1/p}(\varrho)\mathrm{d}\varrho = R^{1/p}(t)z'''(t)\phi_0(t).$$

That is,

$$z''(t) \geq -R^{1/p}(t)z'''(t)\phi_0(t). \tag{43}$$

Hence,

$$\left(\frac{z''(t)}{\phi_0(t)}\right)' = \frac{R^{1/p}(t)\phi_0(t)z'''(t) + z''(t)}{R^{1/p}(t)\phi_0^2(t)} \geq 0.$$

Since $z''(t)/\phi_0(t)$ is increasing, then

$$-z'(t) \geq \int_t^\infty \frac{z''(\varrho)}{\phi_0(\varrho)}\phi_0(\varrho)\mathrm{d}\varrho \geq \frac{z''(t)}{\phi_0(t)}\int_t^\infty \phi_0(\varrho)\mathrm{d}\varrho = \frac{z''(t)}{\phi_0(t)}\phi_1(\varrho).$$

That is,

$$z'(t) \leq -\frac{z''(t)}{\phi_0(t)}\phi_1(\varrho), \tag{44}$$

which implies

$$\left(\frac{z'(t)}{\phi_1(t)}\right)' = \frac{\phi_1(t)z''(t) + \phi_0(t)z'(t)}{\phi_1^2(t)} \leq 0.$$

By repeating the same steps for the decreasing function $z'(t)/\phi_1(t)$, we have

$$\left(\frac{z(t)}{\phi_2(t)}\right)' = \frac{\phi_2(t)z'(t) + \phi_1(t)z(t)}{\phi_2^2(t)} \geq 0.$$

($\mathrm{N}_{1,2}$) From the monotonicity of $R^{1/p}(\varrho)z'''(\varrho)$, we obtain that

$$R^{1/p}(t)z'''(t)\phi_0(t) \geq \int_t^\infty \frac{R^{1/p}(\varrho)z'''(\varrho)}{R^{1/p}(\varrho)}\mathrm{d}\varrho \geq -z''(t),$$

or, equivalently,

$$z''(t) \geq -R^{1/p}(t)z'''(t)\phi_0(t).$$

Integrating the last inequality from t to ∞, we have

$$\begin{aligned}
-z'(t) &\geq -\int_t^\infty R^{1/p}(\varrho)z'''(\varrho)\phi_0(\varrho)\mathrm{d}\varrho \\
&\geq -R^{1/p}(t)z'''(t)\int_t^\infty \phi_0(\varrho)\mathrm{d}\varrho \\
&\geq -R^{1/p}(t)z'''(t)\phi_1(t),
\end{aligned}$$

i.e.,

$$z'(t) \leq R^{1/p}(t)z'''(t)\phi_1(t).$$

Again integrating the last inequality from t to ∞ implies that

$$z(t) \geq -R^{1/p}(t)z'(t)\phi_2(t),$$

and this completes the proof. □

Lemma 15. *Let $x \in \Omega_3$ and (5) holds. Then,*
($\mathrm{N}_{2,1}$) $x(t) > \mathrm{c}_1(t,n)z(t)$;
($\mathrm{N}_{2,2}$) $\left(R(t)(-z'''(t))^p\right)' - Q(t)z^\beta(t) \geq 0$.

Proof. Let $x \in \Omega_3$; then, there exists a $t_1 \geq t_0$ such that $x(h(t)) > 0$ and $x(\kappa(t)) > 0$ for $t \geq t_1$.
$(\mathrm{N}_{2,1})$ In view of the (7) and the increasing monotonicity of $z(t)/\phi_2(t)$, we have

$$z\left(h^{[2i+1]}(t)\right) \leq \frac{\phi_2\left(h^{[2i+1]}(t)\right)}{\phi_2\left(h^{[2i]}(t)\right)} z\left(h^{[2i]}(t)\right),$$

for $h^{[2i]}(t) \geq h^{[2i+1]}(t)$. Substituting the previous inequality into (7), we have

$$x(t) > \sum_{i=0}^{n}\left(\prod_{k=0}^{2i} \mathrm{c}\left(h^{[k]}(t)\right)\right)\left[\frac{1}{\mathrm{c}\left(h^{[2i]}(t)\right)} - \frac{\phi_2\left(h^{[2i+1]}(t)\right)}{\phi_2\left(h^{[2i]}(t)\right)}\right] z\left(h^{[2i]}(t)\right).$$

Since $z'(t) < 0$, and $h^{[2i]}(t) \leq t$, the previous inequality becomes

$$\begin{aligned} x(t) &\geq \sum_{i=0}^{n}\left(\prod_{k=0}^{2i} \mathrm{c}\left(h^{[k]}(t)\right)\right)\left[\frac{1}{\mathrm{c}\left(h^{[2i]}(t)\right)} - \frac{\phi_2\left(h^{[2i+1]}(t)\right)}{\phi_2\left(h^{[2i]}(t)\right)}\right] z(t) \\ &= \mathrm{c}_1(t;n) z(t). \end{aligned} \tag{45}$$

$(\mathrm{N}_{2,2})$ By combining (45) and (42), with $z'''(t) < 0$, we thus deduce that

$$\begin{aligned} \left(R(t)(-z'''(t))^p\right)' &= E(t)q(t)x^\beta(\kappa(t)) \\ &\geq E(t)q(t)\mathrm{c}_1^\beta(\kappa(t);n)z^\beta(\kappa(t)) \\ &= Q(t)z^\beta(\kappa(t)) \geq Q(t)z^\beta(t), \end{aligned}$$

and this completes the proof. □

Now, let us define another auxiliary function $v(t)$ by

$$v(t) := \frac{R(t)(-z'''(t))^p}{(z''(t))^\beta},\ t \geq t_1, \tag{46}$$

then we can have the following improved lemma:

Lemma 16. *Let $x \in \Omega_3$ and (5) holds. Then*
$(\mathrm{N}_{3,1})$ *$v(t)\phi_0^\mu(t)$ is bounded;*
$(\mathrm{N}_{3,2})$ *$v'(t) \geq Q(t)\phi_2^\beta(t)/\phi_0^\beta(t) + \beta m R^{-1/p}(t) v^{(\mu+1)/\mu}(t)$,*
where m is a positive constant and $\mu = \max\{p, \beta\}$.

Proof. Let $x \in \Omega_3$; then, there exists a $t_1 \geq t_0$ such that $x(h(t)) > 0$ and $x(\kappa(t)) > 0$ for $t \geq t_1$.

$(\mathrm{N}_{3,1})$ By Lemma 14, we have $\left(R(t)(-z'''(t))^p\right)' \geq 0$, which implies that $R(t)(-z'''(t))^p$ is nondecreasing. From (43), we have

$$(z''(t))^p \geq R(t)(-z'''(t))^p \phi_0^p(t) = (z''(t))^\beta v(t)\phi_0^p(t), \tag{47}$$

i.e.,

$$(z''(t))^{p-\beta} \geq v(t)\phi_0^p(t),\ t \geq t_1. \tag{48}$$

If $p > \beta$, using $z''' < 0$ in (48), we then find that the positive function $v(t)\phi_2^p(t)$ is bounded.

Now, if $\beta \geq p$, and once again using (43), we obtain

$$(z''(t))^{\beta} \geq \left(R^{1/p}(t)(-z'''(t))\right)^{\beta-p+p} \phi_0^{\beta}(t), \tag{49}$$

which implies that

$$\left[R^{1/p}(t)(-z'''(t))\right]^{p-\beta} \geq \frac{R(t)(-z'''(t))^{p}}{(z''(t))^{\beta}} \phi_0^{\beta}(t) = v(t)\phi_0^{\beta}(t).$$

Since $\left[R^{1/p}(t)(-z'''(t))\right]^{p-\beta}$ is decreasing, then $v(t)\phi_0^{\beta}(t)$ is bounded. Therefore, the function $v(t)\phi_0^{\mu}(t)$ is bounded, where $\mu = \max\{p, \beta\}$.

$(\mathrm{N}_{3,2})$ In view of the definitions of $v(t)$ and $(\mathrm{N}_{1,2})$, we have

$$\begin{aligned} v'(t) &= \frac{\left(R(t)(-z'''(t))^{p}\right)'}{(z''(t))^{\beta}} + \beta \frac{R(t)(-z'''(t))^{p+1}}{(z''(t))^{\beta+1}} \\ &\geq Q(t)\frac{z^{\beta}(t)}{(z''(t))^{\beta}} + \frac{\beta}{R^{1/p}(t)} v^{(p+1)/p}(t)\left(z''(t)\right)^{(\beta-1)/p}. \end{aligned} \tag{50}$$

Using (44), we obtain

$$\frac{z(t)}{z''(t)} \geq \frac{\phi_2(t)}{\phi_1(t)}. \tag{51}$$

Substituting (51) into (50), then

$$v'(t) \geq Q(t)\left(\frac{\phi_2(t)}{\phi_0(t)}\right)^{\beta} + \frac{\beta}{R^{1/p}(t)} v^{(p+1)/p}(t)\left(z''(t)\right)^{(\beta-p)/p}.$$

If $p > \beta$, and taking into account that $z'''(t) < 0$ for $t \geq t$, then $(z''(t))^{(\beta-p)/p}$ is increasing. By letting $m_1 = (z''(t))^{(\beta-p)/p}$ (if $p = \beta$, then $m_1 = 1$), and the above inequality becomes

$$v'(t) \geq Q(t)\left(\frac{\phi_2(t)}{\phi_0(t)}\right)^{\beta} + \beta m_1 R^{-1/p}(t) v^{1/p+1}(t), \ t \geq t_1. \tag{52}$$

Now, if $\beta \geq p$, we have

$$v'(t) \geq Q(t)\left(\frac{\phi_2(t)}{\phi_0(t)}\right)^{\beta} + \beta R^{-1/\beta}(t)\left(-z'''(t)\right)^{(\beta-p)/\beta} v^{(\beta+1)/\beta}(t). \tag{53}$$

Since $\left(R^{1/p}(t)(-z'''(t))\right)^{(\beta-p)/\beta}$ is an increasing function, then, from (53), we obtain

$$\begin{aligned} v'(t) &\geq Q(t)\left(\frac{\phi_2(t)}{\phi_0(t)}\right)^{\beta} + \beta R^{-1/p}(t)\left(R^{1/p}(t)(-z'''(t))\right)^{(\beta-p)/\beta} v^{(\beta+1)/\beta}(t) \\ &\geq Q(t)\left(\frac{\phi_2(t)}{\phi_0(t)}\right)^{\beta} + \beta m_2 R^{-1/p}(t) v^{(\beta+1)/\beta}(t), \ t \geq t_1 \geq t_0, \end{aligned} \tag{54}$$

where $m_2 = \left(R^{1/p}(t_1)(-z'''(t_1))\right)^{1-p/\beta}$ (if $p = \beta$, then $m_2 = 1$).

Combining (52) and (54) yields

$$v'(t) \geq Q(t)\left(\frac{\phi_2(t)}{\phi_0(t)}\right)^{\beta} + \beta m R^{-1/p}(t) v^{(\mu+1)/\mu}(t), \ t \geq t_1, \tag{55}$$

where $\mu = \max\{p, \beta\}$, and

$$m = \begin{cases} 1, & p = \beta \\ \text{const} > 0, & p \neq \beta. \end{cases}$$

And this completes the proof. □

Lemma 17. *Assume that (5) and* $c(t) < \phi_2(t)/\phi_2(h(t))$ *hold. If*

$$\limsup_{t\to\infty} \int_{t_3}^{t} \left(\phi_0^{\mu}(\varrho) Q(\varrho) \left(\frac{\phi_2(\varrho)}{\phi_0(\varrho)} \right)^{\beta} - \frac{L}{\phi_0(\varrho) R^{1/p}(\varrho)} \right) \mathrm{d}\varrho = \infty, \tag{56}$$

then $\Omega_3 = \varnothing$.

Proof. Suppose the contrary that $x \in \Omega_3$, i.e., there exists a $t_1 \geq t_0$ such that $x(h(t)) > 0$ and $x(\kappa(t))$ for all $t \geq t_1$. Considering the fact that $z(t) \geq x(t) > 0$ for $t \geq t_1$ and (41), we have

$$\left(R(t) |z'''(t)|^{p-1} z'''(t) \right)' = -E(t) q(t) x^{\beta}(\kappa(t)) \leq 0,$$

which implies that $R(t)|z'''(t)|^{p-1}z'''(t)$ is nonincreasing. Since $z'''(t) < 0$, by Lemma 15, we obtain

$$\left(R(t) \left(-z'''(t) \right)^{p} \right)' - Q(t) z^{\beta}(t) \geq 0, \ t \geq t_1.$$

Let $v(t)$ be defined by (46) for $t \geq t_2 \geq t_1$. It then follows that $v(t) > 0$ for all $t \geq t_2$. From Lemma 16, we have

$$v'(t) \geq Q(t) \phi_2^{\beta}(t) / \phi_0^{\beta}(t) + \beta m R^{-1/p}(t) v^{(\mu+1)/\mu}(t), \ t \geq t_2. \tag{57}$$

Multiplying (57) by $\phi_0^{\mu}(t)$ and integrating the resulting inequality from $t_3 \geq t_2$ to t, we have

$$\begin{aligned} & \int_{t_3}^{t} \phi_0^{\mu}(\varrho) Q(\varrho) \left(\frac{\phi_2(\varrho)}{\phi_0(\varrho)} \right)^{\beta} \mathrm{d}\varrho \\ \leq & \int_{t_3}^{t} \phi_0^{\mu-1}(\varrho) R^{-1/p}(\varrho) \left[\mu v(\varrho) - \beta m \phi_0(\varrho) v^{(\mu+1)/\mu}(\varrho) \right] \mathrm{d}\varrho \\ & + \phi_2^{\mu}(t) v(t). \end{aligned} \tag{58}$$

Using Lemma 3, where $B = \mu$, $A = \beta m_1 \phi_0(t)$ and $u = v(t)$, we have

$$\begin{aligned} \mu v(t) - \beta m \phi_0(t) v^{(\mu+1)/\mu}(t) & \leq \frac{\mu^{\mu}}{(\mu+1)^{\mu+1}} \frac{\mu^{\mu+1}}{\beta^{\mu} m^{\mu} \phi_2^{\mu}(t)} \\ & = \left(\frac{\mu}{\mu+1} \right)^{\mu+1} \left(\frac{\mu}{\beta m} \right)^{\mu} \frac{1}{\phi_0^{\mu}(t)} \\ & = L \frac{1}{\phi_0^{\mu}(t)}, \end{aligned}$$

which, with (58), gives

$$\int_{t_3}^{t} \left(\phi_0^{\mu}(\varrho) Q(\varrho) \left(\frac{\phi_2(\varrho)}{\phi_0(\varrho)} \right)^{\beta} - \frac{L}{\phi_0(\varrho) R^{1/p}(\varrho)} \right) \mathrm{d}\varrho \leq \phi_0^{\mu}(t) v(t),$$

where

$$L=\begin{cases}\left(\frac{\mu}{\mu+1}\right)^{\mu+1}\left(\frac{\mu}{\beta m}\right)^{\mu}, & p\neq\beta,\\ \left(\frac{p}{p+1}\right)^{p+1}, & p=\beta.\end{cases}$$

From Lemma 16, we see that $\phi_0^{\mu}(t)v(t)$ is bounded. Letting $t\to\infty$ in the above inequality, we obtain a contradiction with (56). And this completes the proof. $\square$

Lemma 18. *Assume that (5) and* $\mathrm{c}(t)<\phi_2(t)/\phi_2(h(t))$ *hold. If*

$$\liminf_{t\to\infty}\phi_0^{\mu+1}(t)R^{1/p}(t)Q(t)\left(\frac{\phi_2(t)}{\phi_0(t)}\right)^{\beta}>L, \tag{59}$$

then $\Omega_3=\varnothing$.

Proof. Suppose that (59) holds; then, for any $\varepsilon>0$, there exists a sufficiently large $t\geq t_0$ such that

$$\phi_0^{\mu+1}(t)Q(t)\left(\frac{\phi_2(\varrho)}{\phi_0(\varrho)}\right)^{\beta}>\frac{L-\varepsilon}{\phi_0(t)R^{1/p}(t)}.$$

Integrating this inequality from t to t, we then obtain

$$\begin{aligned}\int_{t_3}^{t}\left(\phi_0^{\mu}(\varrho)Q(\varrho)\left(\frac{\phi_2(\varrho)}{\phi_0(\varrho)}\right)^{\beta}-\frac{L}{\phi_0(\varrho)R^{1/p}(\varrho)}\right)\mathrm{d}\varrho &> -\varepsilon\int_{t_3}^{t}\frac{1}{\phi_0(t)R^{1/p}(t)}\mathrm{d}\varrho\\ &= \varepsilon\left(\ln\frac{1}{\phi_0(t)}-\ln\frac{1}{\phi_0(t)}\right).\end{aligned}$$

Letting $t\to\infty$ in the above inequality, we find that (56) holds. The proof is complete. $\square$

4. Applications on Oscillation Theorems

This section extends the insights from the previous section to develop novel criteria for examining the oscillation of all solutions within (1). By combining the conditions established earlier that exclude the positive solutions in all cases (C_1) (C_2) and (C_3), we can formulate the following oscillation criteria for (1):

Theorem 1. *Let* $p\geq 1$. *Assume that (56), (13), and (25) hold. Then, (1) is oscillatory.*

Proof. Suppose that x is a solution to (1) that eventually becomes positive. According to Lemma 4, there exist three possible cases for the behavior of z and its derivatives. By applying Lemmas 17, 8, and 12, it becomes clear that under conditions (56), (13), and (25), there are no positive solutions to (1) that satisfy the cases (C_1), (C_2), and (C_3). Consequently, we can confidently assert that our proof is now complete $\square$

Theorem 2. *Let* $p\geq 1$. *Assume that (59), (22) and (40) hold. Then (1) is oscillatory.*

Proof. The proof of this theorem follows the same method as the proof of the Theorem (1), and as such, it was omitted. $\square$

Example 1. *Consider the nonlinear NDDE:*

$$\left(t^{p}\left|z'''(t)\right|^{p-1}z'''(t)\right)'+\frac{3p}{t^{1-p}}\left|z'''(t)\right|^{p-1}z'''(t)+\frac{q_0}{t^{2p+1}}\left|x(\kappa_0 t)\right|^{p-1}x(\kappa(t)=0, \tag{60}$$

where

$$z(t)=x(t)+\mathrm{c}_0x(h_0t),$$

$t \geq 1$, $p \geq 1$, $0 \leq c_0 < 1$, $h_0, \kappa_0 \in (0,1)$, and $q_0 > 0$. By comparing this equation with (1), we observe that $\beta = p \geq 1$, $a(t) = t^p$, $q(t) = q_0/t^{2p+1}$, $c(t) = c_0$, $\kappa(t) = \kappa_0 t$,, and $h(t) = h_0 t$. It is easy to find that

$$E(t) = t^{3p},\ R(t) = t^{4p},$$

$$\phi_0(t) = \frac{1}{3t^3},\ \phi_1(t) = \frac{1}{6t^2},\ \phi_2(t) = \frac{1}{6t},$$

$$c_1(t;n) = (h_0 - c_0)\sum_{i=0}^{n} c_0^{2i},$$

$$c_2(t;n) := (1 - c_0)\sum_{i=0}^{n} c_0^{2i} h_0^{4i/\epsilon_0},$$

$$Q(t) := E(t)q(t)c_1^p = q_0 t^{p-1} c_1^p,$$

$$Q_1(t) := q_0 t^{p-1}(1 - c_0)^p,$$

and

$$Q_2(t) := q_0 t^{p-1} c_2^p.$$

Condition (56) yields

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{t_3}^{t}\left(\phi_0^{\mu}(\varrho)Q(\varrho)\left(\frac{\phi_2(\varrho)}{\phi_0(\varrho)}\right)^{\beta} - \frac{L}{\phi_0(\varrho)R^{1/p}(\varrho)}\right)d\varrho \\
=\ &\limsup_{t\to\infty}\int_{t_3}^{t}\left(\frac{1}{3^p\varrho^{3p}}q_0\varrho^{p-1}c_1^p\frac{3^p\varrho^{3p}}{6^p\varrho^p} - \frac{3\varrho^3 L}{\varrho^4}\right)d\varrho \\
=\ &\limsup_{t\to\infty}\int_{t_3}^{t}\left(\frac{q_0}{6^p}c_1^p - 3L\right)\frac{1}{\varrho}d\varrho \\
=\ &\left(\frac{q_0}{6^p}c_1^p - 3L\right)\limsup_{t\to\infty}\ln\frac{t}{t_3} = \infty,
\end{aligned}$$

which holds true when

$$q_0 > 3\left(\frac{6}{c_1}\right)^p L. \tag{61}$$

Condition (13) with $\xi(t) = t^p$ results in

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{t_3}^{t}\left(\xi(\varrho)Q_1(\varrho) - \frac{2^v}{(v+1)^{v+1}}\frac{R(\theta(\varrho))(\xi'(\varrho))^{v+1}}{(K\lambda_1\xi(\varrho)\kappa'(\varrho)\kappa^2(\varrho))^v}\right)d\varrho \\
=\ &\limsup_{t\to\infty}\int_{t_3}^{t}\left(\varrho^p q_0\varrho^{p-1}(1-c_0)^p - \frac{1}{(p+1)^{p+1}}\frac{1}{(K\lambda_1)^p}\frac{\kappa_0^{4p}\varrho^{4p}p^{p+1}\varrho^{p^2-1}}{\kappa_0^{3p}\varrho^{2p}\varrho^{p^2}}\right)d\varrho \\
=\ &\limsup_{t\to\infty}\int_{t_3}^{t}\left(q_0(1-c_0)^p - \frac{1}{(p+1)^{p+1}}\left(\frac{2\kappa_0}{K\lambda_1}\right)^p\right)\varrho^{2p-1}d\varrho \\
=\ &\frac{1}{2p}\left(q_0(1-c_0)^p - \frac{1}{(p+1)^{p+1}}\left(\frac{2\kappa_0}{K\lambda_1}\right)^p\right)\limsup_{t\to\infty} t^{2p} = \infty,
\end{aligned}$$

which is satisfied when:

$$q_0 > \frac{1}{(p+1)^{p+1}}\left(\frac{2\kappa_0}{K\lambda_1(1-c_0)}\right)^p. \tag{62}$$

Condition (25) with $\widetilde{\xi}(t) = t^p$ gives

$$\begin{aligned}\Psi(t) &= \widetilde{\xi}(t)Q_2(t)\zeta(t)\left(\frac{\lambda_2}{2}\kappa^2(t)\right)^\beta - \frac{(p-1)\widetilde{\xi}(t)}{R^{1/p}(t)\phi_0^{p+1}(t)} \\ &= t^p q_0 t^{p-1} c_2^p \left(\frac{\lambda_2}{2}\right)^p \kappa_0^{2p} t^{2p} - \frac{(p-1)3^{p+1}t^{3p+3}t^p}{t^4} \\ &= \left(q_0 c_2^p \left(\frac{\lambda_2}{2}\right)^p \kappa_0^{2p} - (p-1)3^{p+1}\right) t^{4p-1},\end{aligned}$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\int_{t_1}^t \left(\Psi(\varrho) - \frac{R(\varrho)\widetilde{\xi}(\varrho)}{(p+1)^{p+1}}\left(\frac{\widetilde{\xi}'(\varrho)}{\widetilde{\xi}(\varrho)} + \frac{(1+p)}{R^{1/p}(\varrho)\phi_0(\varrho)}\right)^{p+1}\right)\mathrm{d}\varrho \\ =\ & \limsup_{t\to\infty}\int_{t_1}^t \left(\left(q_0 c_2^p\left(\frac{\lambda_2}{2}\right)^p \kappa_0^{2p} - (p-1)3^{p+1}\right) - \left(\frac{p+3(1+p)}{p+1}\right)^{p+1}\right)\varrho^{4p-1}\mathrm{d}\varrho \\ =\ & \frac{1}{4p}\left(\left(q_0 c_2^p\left(\frac{\lambda_2}{2}\right)^p \kappa_0^{2p} - (p-1)3^{p+1}\right) - \left(\frac{p+3(1+p)}{p+1}\right)^{p+1}\right)\limsup_{t\to\infty} t^{4p} = \infty,\end{aligned}$$

which holds when

$$q_0 > \left(\frac{2}{\lambda_2 c_2 \kappa_0^2}\right)^p \left[\left(\frac{p+3(1+p)}{p+1}\right)^{p+1} + (p-1)3^{p+1}\right]. \tag{63}$$

By using Theorem 1, we have that (60) is oscillatory if the conditions (61), (62), and (63) hold. This can be verified by assigning specific values to Equation (60).

Remark 1. *By taking $p = 1$, $c_0 = 0.5$, $h_0 = 0.8$, $\kappa_0 = 0.7$, and $q_0 = 255$, (60) becomes*

$$\left(t(x(t) + 0.5x(0.8t))'''\right)' + 3(x(t) + 0.5x(0.8t))''' + \frac{255}{t^3}x(0.7t) = 0. \tag{64}$$

By comparing this equation with (1), we find that $p = \beta = 1$, $a(t) = t$, $q(t) = q_0/t^3$, $c(t) = 1/2$, $\kappa(t) = t/3$ and $h(t) = t/4$. Consequently, we can easily deduce:

$$E(t) = t^3,\ R(t) = t^4,$$

$$\phi_0(t) = \frac{1}{3t^3},\ \phi_1(t) = \frac{1}{6t^2},\ \phi_2(t) = \frac{1}{6t},$$

$$c_1(t;10) = (0.8 - 0.5)\sum_{i=0}^{5}(0.8)^{2i} \cong 0.78,$$

$$c_2(t;10) = \sum_{i=0}^{10}\left((0.5)^{2i+1}(0.8)^{4i/\epsilon_0}\right) = 0.55,\ \textit{where } \epsilon_0 = 0.9,$$

$$Q(t) = 157.5,\ Q_1(t) = 112.5,\ \textit{and } Q_2(t) = 123.8.$$

Condition (59) becomes

$$\begin{aligned}\liminf_{t\to\infty}\phi_0^{\mu+1}(t)R^{1/p}(t)Q(t)\left(\frac{\phi_2(\varrho)}{\phi_0(\varrho)}\right)^\beta &= \liminf_{t\to\infty}\frac{1}{3^2t^6}t^4(157.5)\frac{3t^3}{6t} \\ &= 8.75 > \frac{1}{4}.\end{aligned}$$

Condition (22) leads to

$$\limsup_{t\to\infty}\int_{t_3}^{t} Q_1(t)\mathrm{d}\varrho = \limsup_{t\to\infty}\int_{t_3}^{t} 112.5\mathrm{d}\varrho = \infty.$$

And, condition (40) with $\lambda_2 = 0.5$ *becomes*

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{t_1}^{t}\left(Q_2(\varrho)\zeta(\varrho)\left(\frac{\lambda_2}{2}\kappa^2(\varrho)\right)^{\beta} - \frac{p}{R^{1/p}(\varrho)\phi_0^{p+1}(\varrho)}\right)\mathrm{d}\varrho \\
= &\limsup_{t\to\infty}\int_{t_1}^{t}\left(123.8\left(\frac{0.5}{2}(0.7)^2\varrho^2\right) - \frac{9\varrho^6}{\varrho^4}\right)\mathrm{d}\varrho \\
= &\limsup_{t\to\infty}\int_{t_1}^{t} 6.2\varrho^2\mathrm{d}\varrho = \infty.
\end{aligned}$$

Hence, conditions (59), (22), and (40) are satisfied, and by using Theorem 2, we see that (64) is oscillatory

5. Conclusions

This paper conducted an extensive inquiry into the asymptotic and oscillatory behavior of a specific subclass of fourth-order nonlinear neutral differential equations. Our investigation was particularly centered on their noncanonical form, which was augmented with the introduction of damping terms. The primary goal of this research was to elevate our understanding of the intricate relationships between the solutions of these equations and their corresponding functions. By refining these relationships, we unearthed novel insights into the monotonic properties that govern these solutions. These insights, in turn, enabled us to derive improved conditions and parameters for the analyzed equation. Employing Riccati's technique and the comparison method, this study furnished robust criteria that ensure the presence of oscillatory behavior in the solutions.

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