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Some Sufficient Conditions for Analytic Functions

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Author's contribution

This whole work was carried out by the author HL.

Original Research Article

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ABSTRACT

In this paper introduced some new subclasses of analytic functions in the unit disc. We obtain the sufficient conditions for starlike ness.

Keywords: Analytic function; close-to-convex function; starlike function.

1. INTRODUCTION

Let H denote the class of analytic functions in $U=\{z\in C:|z|<1\}$ and A denote the subclass of H, which consist as functions of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots, z\in U. \tag{1}$$

A function $f(z)\in A$ is consist as starlike of order $\alpha(0\le\alpha<p)$ in U (see [1]), that is, $f(z)\in S^*(\alpha)$, if and only if

$$Re(\frac{zf'(z)}{f(z)})>\alpha,(0\le\alpha<1), z\in U \tag{2}$$

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with $S^*(0) := S^*$.

Similarly, a function $f(z) \in A$ is consist as convex of order $\alpha (0 \le \alpha < 1)$ in U (see [1]), that is, $f(z) \in K(\alpha)$, if and only if

$$Re(1 + \frac{zf''(z)}{f'(z)}) > \alpha, (0 \le \alpha < 1), z \in U \tag{3}$$

with $K(0) = K$.

According to the definitions for the classes $S^*(\alpha)$ and $K(\alpha)$, we know that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$. Marx [2] and Strohhäcker [3] showed that $f(z) \in K(0)$ implies $f(z) \in S^*(1/2)$.

Ozaki [4] and Kaplan [5] investigated the following functions : If $f(z) \in A$ satisfies

$$Re(\frac{f'(z)}{g'(z)}) > 0, z \in U \tag{4}$$

for some convex function $g(z)$, then $f(z)$ is univalent function in U. In the view of Kaplan (see [5]), we say that $f(z)$ satisfying the above inequality is close-to-convex in U, that is, $f(z) \in C(0) := C$.

It is well known that the above definition concerning close-to-convex functions, is equivalent to the following condition:

$$Re(\frac{zf'(z)}{g(z)}) > 0, z \in U \tag{5}$$

for some starlike function $g(z) \in A$.

A function $f(z) \in A$ is consist as close-to-convex of order $\alpha (0 \le \alpha < p)$ in U with respect to $g(z)$, that is, $f(z) \in C(\alpha)$, if and only if

$$Re(\frac{zf'(z)}{g(z)}) > \alpha, z \in U \tag{6}$$

for some real $\alpha (0 \le \alpha < 1)$ and for some starlike function $g(z) \in A$.

In this article, using the conditions and lemmas which were different from the reference [6], the author introduced the subclasses of close-to-convex functions and obtained some sufficient conditions and extended some earlier works.

2. MAIN RESULTS

To prove our results, we will need the following lemmas:

Lemma 2.1. (see [7])Let $p(z)=1+c_1z+c_2z^2+\cdots$ be analytic in the unit disc U and $\alpha(0<\alpha\le 1/2)$ be a positive real number. Then suppose that there exists a point $z_0\in U$ such that

$$Re\{p(z)\}>\alpha \, for \, |z|<|z_0| \tag{7}$$

and

$$Re\{p(z_0)\}=p(z_0)=\alpha. \tag{8}$$

Then we have

$$\frac{z_0p'(z_0)}{p(z_0)}\le -k(1-\alpha) \tag{9}$$

where $k\ge 1$ is a real number.

Lemma 2.2. Let $p(z)=1+c_1z+c_2z^2+\cdots$ be analytic in the unit disc U and $\alpha(0<\alpha\le 1/2)$ be a positive real number. Suppose also that for arbitrary $r(0<r<1)$, fulfills this condition

$$\min_{|z|\le r} Re\{p(z)\}=\min_{|z|\le r} |p(z)| \tag{10}$$

and

$$Re(\frac{zp'(z)}{p(z)})>\alpha-1, z\in U. \tag{11}$$

Then we have

$$Re\{p(z)\}>\alpha, z\in U. \tag{12}$$

Proof. Suppose that there exists a point $z_0\in U$ such that

$$Re\{p(z)\}>\alpha \, for \, |z|<|z_0| \tag{13}$$

and

$$Re\{p(z_0)\} = \alpha, 0 < \alpha \le \frac{1}{2}. \tag{14}$$

From the hypothesis of Lemma 2.2, then we have

$$Re\{p(z_0)\} = p(z_0) = \alpha, 0 < \alpha \le \frac{1}{2}. \tag{15}$$

From Lemma 2.1, then we have

$$Re(\frac{z_0 p'(z_0)}{p(z_0)}) \le \alpha - 1, 0 < \alpha \le \frac{1}{2}. \tag{16}$$

This contradicts the hypothesis (11) of Lemma 2.2 and it completes the proof of Lemma 2.2.

By using Lemma 2.2, we first prove the following Theorem.

Theorem 2.1. Let $f(z) \in A$, and $\alpha(0 < \alpha \le 1/2)$ be a positive real number. Suppose that there exists a starlike function $g(z)$ such that

$$\min_{|z|\le r} Re(\frac{zf'(z)}{g(z)}) = \min_{|z|\le r} |\frac{zf'(z)}{g(z)}| \tag{17}$$

for arbitrary $r(0 < r < 1)$, and

$$1 + Re\frac{zf''(z)}{f'(z)} > Re\frac{zg'(z)}{g(z)} + \alpha - 1, 0 < \alpha \le \frac{1}{2}, z \in U. \tag{18}$$

Then we have $f(z) \in C(\alpha)$.

Proof. Let

$$p(z) = \frac{zf'(z)}{g(z)}, \tag{19}$$

then $p(z)$ is analytic in U and $p(0) = 1$. Now using (19), it follows that

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} = \frac{zp'(z)}{p(z)}. \tag{20}$$

By Lemma 2.2 and the hypothesis (17), (18) in Theorem 2.1, we obtain

$$Re(\frac{zf'(z)}{g(z)}) > \alpha, 0 < \alpha \le \frac{1}{2}, z \in U. \quad (21)$$

Therefore proof of the Theorem 2.1 is completed.

Lemma 2.3. (see [7])Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in the unit disc U and $\alpha(1/2 < \alpha < 1)$ be a positive real number. Then suppose that there exists a point $z_0 \in U$ such that

$$Re\{p(z)\} > \alpha \, for \, |z| < |z_0| \quad (22)$$

and

$$Re\{p(z_0)\} = p(z_0) = \alpha. \quad (23)$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} \le -\frac{k}{2}(2-\alpha) \quad (24)$$

where $k \ge 1$ is a real number.

Lemma 2.4. Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in the unit disc U and $\alpha(1/2 < \alpha < 1)$ be a positive real number. Suppose also that for arbitrary $r(0 < r < 1)$, fulfills this condition

$$\min_{|z| \le r} Re \quad p(z) = \min_{|z| \le r} |p(z)| \quad (25)$$

and

$$Re(\frac{zp'(z)}{p(z)}) > \frac{\alpha}{2} - 1, z \in U. \quad (26)$$

Then we have

$$Re\{p(z)\} > \alpha, z \in U. \quad (27)$$

Proof. Suppose that there exists a point $z_0 \in U$ such that

$$Re\{p(z)\} > \alpha \, for \, |z| < |z_0| \quad (28)$$

and

$$Re\{p(z_0)\} = \alpha, \frac{1}{2} < \alpha < 1. \tag{29}$$

By the hypothesis of Lemma 2.4, we have

$$Re\{p(z_0)\} = p(z_0) = \alpha, \frac{1}{2} < \alpha < 1. \tag{30}$$

Making use of Lemma 2.3, then we have

$$Re(\frac{z_0 p'(z_0)}{p(z_0)}) \leq \frac{\alpha}{2} - 1, \frac{1}{2} < \alpha < 1. \tag{31}$$

This contradicts the hypothesis (26) of Lemma 2.4 and it completes the proof of Lemma 2.4. Making use of Lemma 2.4, we can prove the following Theorem.

Theorem 2.2. Let $f(z) \in A$, and $\alpha(1/2 < \alpha < 1)$ be a positive real number. Suppose that there exists a starlike function $g(z)$ such that

$$\min_{|z| \leq r} Re(\frac{zf'(z)}{g(z)}) = \min_{|z| \leq r} |\frac{zf'(z)}{g(z)}| \tag{32}$$

for arbitrary $r(0 < r < 1)$, and

$$1 + Re\frac{zf''(z)}{f'(z)} > Re\frac{zg'(z)}{g(z)} + \frac{\alpha}{2} - 1, z \in U. \tag{33}$$

Then we have $f(z) \in C(\alpha)$.

Proof. Let

$$p(z) = \frac{zf'(z)}{g(z)}, \tag{34}$$

then $p(z)$ is analytic in U and $p(0) = 1$. Now using (34), it follows that

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} = \frac{zp'(z)}{p(z)}. \tag{35}$$

By Lemma 2.4 and the hypothesis (32), (33) in Theorem 2.2, we obtain

$$Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha, \frac{1}{2} < \alpha < 1, z \in U. \tag{36}$$

Therefore proof of the Theorem 2.2 is completed.

3. CONCLUSION

In this work studied some sufficient conditions for starlike ness of the new subclasses of analytic functions.

COMPETING INTERESTS

Author has declared that no competing interests exist.

REFERENCES

1. Robertson MS. On the theory of univalent functions. Ann of Math. 1936;37:374-408.
2. Marx A. Studies of simple pictures. Math Ann. 1932;33(107):40-67.
3. Strohhäcker E. Contributions to the theory of simple functions. Math Zeit. 1933;37:356-380.
4. Ozaki S. On the theory of multivalent functions. Sci Rep Tokyo Bunrika Daigo. 1935;2:167-188.
5. Kaplan W. Close-to-convex schlicht functions. Michigan Math J. 1952;1:169-185.
6. Nunokawa M, Aydogan M, Kuroki K, Yildiz I, Owa S. Some properties concerning close-to-convexity of certain analytic functions. Journal of Inequalities and Applications. 2012;2012:245. DOI: 10.1186/1029-242X-2012-245.
7. Sokol J, Nunokawa M. On some sufficient conditions for univalence and starlikeness. Journal of Inequalities and Applications. 2012;2012:282. DOI: 10.1186/1029-242X-2012-282.

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