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Asymptotic Solutions for the Fifth Order Critically Damped Nonlinear Systems in the Case for Small Equal Eigenvalues

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Abstract

This article examines a fifth order critically damped nonlinearsystem in the case of small equal eigenvalues and tries to find out an asymptotic solution. This paper suggests that the solutions obtained by the perturbation techniques based on modified *Krylov-Bogoliubov-Mitropoloskii* (KBM) method is consistent with the numerical solutions obtained by the fourth order *Runge-Kutta* method.

Keywords

KBM, Eigenvalues, Critically Damped System, Nonlinearity, Asymptotic Solution, *Runge-Kutta* Method

1. Introduction

The *Krylov-Bogoliubov-Mitropoloskii* ([1] [2]) method, known as KBM method, is one of the most used methods for analysing nonlinear oscillatory and non-oscillatory differential systems with small nonlinearities. Krylov and Bogoliubov [1] first developed this method to find the periodic solutions of second order nonlinear differential systems with small nonlinearities. However, the method was later improved and justified mathematically by Bogoliubov and Mitropolskii [2]. It was then extended by Popov [3] to damped oscillatory nonlinear systems. The Popov results were rediscovered by Mendelson [4] because of the physical importance of the damped oscillatory systems. In the meantime, Murty *et al*. [5] developed an asymptotic method based on the theory of Bogoliubov to obtain the response of over damped nonlinear systems. Later, Murty [6] offered a unified KBM method, which was capable to cover the damped and over-damped cases. Sattar [7] also examined an asymptotic solution for a second order critically damped nonlinear system. Alam [8] proposed a new asymptotic solution for

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both over-damped and critically damped nonlinear systems. Akbar *et al.* [9] propounded an asymptotic method for fourth order over-damped nonlinear systems, which was straightforward as well as easier than the method put forward by Murty *et al.* [5]. Later, Akbar *et al.* [10] extended the method for fourth order damped oscillatory systems. Akbar *et al.* [11] also suggested a technique for obtaining over-damped solutions of *n*-th order nonlinear differential systems. Recently, Rahaman and Rahman [12] have found analytical approximate solutions of fifth order more critically damped systems in the case of smaller triply repeated roots. Besides, Rahaman and Kawser [13] have also proposed asymptotic solutions of fifth order critically damped nonlinear systems with pair wise equal eigenvalues and another is distinct. Further, Islam *et al.* [14] suggested an asymptotic method of Krylov-Bogoliubov-Mitropolskii for fifth order critically damped nonlinear systems. Furthermore, Rahaman and Kawser [15] expounded analytical approximate solutions of fifth order more critically damped nonlinear systems.

This study seeks to find solutions of fifth order critically damped nonlinear systems where two of the eigenvalues are equal and smaller than the other three distinct eigenvalues. This paper shows that the obtained perturbation results show good coincidence with the numerical results for different sets of initial conditions and eigenvalues.

2. The Method

Consider a fifth order weakly nonlinear ordinary differential system

$$x^{(v)}+k_1x^{(iv)}+k_2\dddot{x}+k_3\ddot{x}+k_4\dot{x}+k_5x=-\varepsilon f\left(x^{(iv)},\dddot{x},\ddot{x},\dot{x},x\right) \tag{1}$$

where $x^{(v)}$ and $x^{(iv)}$ denote the fifth and fourth derivatives respectively and over dots represent the first, second and third derivatives of *x* with respect to *t*; k_1,k_2,k_3,k_4,k_5 are constants, ε is a sufficiently small and positive parameter and $f\left(x^{(iv)},\dddot{x},\ddot{x},\dot{x},x\right)$ is the given nonlinear function. Let us choose that the characteristic equation of the linear equation of (1) has five eigenvalues, where two of the eigenvalues are equal and other three are distinct. Suppose the eigenvalues are $-\lambda,-\lambda,-\mu,-\nu$ and η.

When $\varepsilon=0$, the solution of the corresponding linear equation of (1) is

$$x(t,0)=(a_0+b_0t)e^{-\lambda t}+c_0e^{-\mu t}+d_0e^{-\nu t}+h_0e^{-\eta t} \tag{2}$$

where a_0,b_0,c_0,d_0 and h_0 are integral constants.

When $\varepsilon\neq 0$, following Alom [16], an asymptotic solution of (1) is found in the form

$$x(t,\varepsilon)=(a+bt)e^{-\lambda t}+ce^{-\mu t}+de^{-\nu t}+he^{-\eta t}+\varepsilon u_1(a,b,c,d,h,t)+\cdots \tag{3}$$

where a,b,c,d and *h* are functions of *t* and they satisfy the first order differential equations

$$\begin{aligned}\dot{a}(t)&=\varepsilon A_1(a,b,c,d,h,t)+\cdots\\ \dot{b}(t)&=\varepsilon B_1(a,b,c,d,h,t)+\cdots\\ \dot{c}(t)&=\varepsilon C_1(a,b,c,d,h,t)+\cdots\\ \dot{d}(t)&=\varepsilon D_1(a,b,c,d,h,t)+\cdots\\ \dot{h}(t)&=\varepsilon H_1(a,b,c,d,h,t)+\cdots\end{aligned} \tag{4}$$

In order to determine the unknown functions A_1,B_1,C_1,D_1 and H_1 we differentiate the proposed solution (3) fifth times with respect to *t*, substituting the value of *x* and the derivatives $\dot{x},\ddot{x},\dddot{x},x^{(iv)},x^{(v)}$ in the original equation (1), utilizing the relation presented in (4) and finally equating the coefficients of ε, we obtain

$$\begin{aligned}&e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\nu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial A_1}{\partial t}+t\frac{\partial B_1}{\partial t}+2B_1\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2\left(\frac{\partial}{\partial t}+\nu-\mu\right)\left(\frac{\partial}{\partial t}+\eta-\mu\right)C_1\\ &+e^{-\nu t}\left(\frac{\partial}{\partial t}+\lambda-\nu\right)^2\left(\frac{\partial}{\partial t}+\mu-\nu\right)\left(\frac{\partial}{\partial t}+\eta-\nu\right)D_1+e^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^2\left(\frac{\partial}{\partial t}+\mu-\eta\right)\left(\frac{\partial}{\partial t}+\nu-\eta\right)H_1\\ &+\left(\frac{\partial}{\partial t}+\lambda\right)^2\left(\frac{\partial}{\partial t}+\mu\right)\left(\frac{\partial}{\partial t}+\nu\right)\left(\frac{\partial}{\partial t}+\eta\right)u_1=-f^{(0)}(a,b,c,d,h,t)\end{aligned} \tag{5}$$

where $f^{(0)}(a,b,c,d,h,t) = f\left(x_0, \dot{x}_0, \ddot{x}_0, \dddot{x}, x^{(iv)}\right)$
and $x_0 = (a+bt)\mathrm{e}^{-\lambda t} + c\mathrm{e}^{-\mu t} + d\mathrm{e}^{-\nu t} + h\mathrm{e}^{-\eta t}$.

In this investigation, we have expanded the function $f^{(0)}$ in the Taylor's series (see also Murty *et al.* [5] for details) about the origin in powers of *t*. Therefore, we obtain

$$f^{(0)} = \sum_{q=0}^{\infty} t^q \sum_{i,j,k,l,m=0}^{\infty} F_{q,m}(a,b,c,d,h)\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t} \tag{6}$$

Here the limits of i,j,k,l and *m* are from 0 to ∞. But for a particular problem they have some definite values. Therefore, using (6) in (5), we obtain

$$\begin{aligned}
&\mathrm{e}^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\nu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial A_1}{\partial t}+t\frac{\partial B_1}{\partial t}+2B_1\right)\\
&+\mathrm{e}^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2\left(\frac{\partial}{\partial t}+\nu-\mu\right)\left(\frac{\partial}{\partial t}+\eta-\mu\right)C_1+\mathrm{e}^{-\nu t}\left(\frac{\partial}{\partial t}+\lambda-\nu\right)^2\left(\frac{\partial}{\partial t}+\mu-\nu\right)\left(\frac{\partial}{\partial t}+\eta-\nu\right)D_1\\
&+\mathrm{e}^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^2\left(\frac{\partial}{\partial t}+\mu-\eta\right)\left(\frac{\partial}{\partial t}+\nu-\eta\right)H_1+\left(\frac{\partial}{\partial t}+\lambda\right)^2\left(\frac{\partial}{\partial t}+\mu\right)\left(\frac{\partial}{\partial t}+\nu\right)\left(\frac{\partial}{\partial t}+\eta\right)u_1\\
&=-\sum_{q=0}^{\infty} t^q \sum_{i,j,k,l,m=0}^{\infty} F_{q,m}(a,b,c,d,h)\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t}
\end{aligned} \tag{7}$$

Following the KBM method, Sattar [7], Alam [17], Alam and Sattar ([18] [19]) imposed the condition that u_1 does not contain the fundamental terms of $f^{(0)}$. Therefore, Equation (7) can be separated in the following way:

$$\begin{aligned}
&\mathrm{e}^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\nu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial A_1}{\partial t}+t\frac{\partial B_1}{\partial t}+2B_1\right)+\mathrm{e}^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2\left(\frac{\partial}{\partial t}+\nu-\mu\right)\left(\frac{\partial}{\partial t}+\eta-\mu\right)C_1\\
&+\mathrm{e}^{-\nu t}\left(\frac{\partial}{\partial t}+\lambda-\nu\right)^2\left(\frac{\partial}{\partial t}+\mu-\nu\right)\left(\frac{\partial}{\partial t}+\eta-\nu\right)D_1+\mathrm{e}^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^2\left(\frac{\partial}{\partial t}+\mu-\eta\right)\left(\frac{\partial}{\partial t}+\nu-\eta\right)H_1\\
&=-\left\{\sum_{i,j,k,l,m=0}^{\infty} F_{0,m}(a,b,c,d,h)\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t}+t\sum_{i,j,k,l,m=0}^{\infty} F_{1,m}(a,b,c,d,h)\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t}\right\}
\end{aligned} \tag{8}$$

$$\left(\frac{\partial}{\partial t}+\lambda\right)^2\left(\frac{\partial}{\partial t}+\mu\right)\left(\frac{\partial}{\partial t}+\nu\right)\left(\frac{\partial}{\partial t}+\eta\right)u_1=-\sum_{q=2}^{\infty} t^q \sum_{i,j,k,l,m=0}^{\infty} F_{q,m}(a,b,c,d,h)\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t} \tag{9}$$

Now, equating the coefficients of t^0 and t^1 from both sides of Equation (8), we obtain

$$\mathrm{e}^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\nu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\frac{\partial B_1}{\partial t}=-\sum_{i,j,k,l,m=0}^{\infty} F_{1,m}(a,b,c,d,h)\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t} \tag{10}$$

$$\begin{aligned}
&\mathrm{e}^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\nu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial A_1}{\partial t}+2B_1\right)\\
&+\mathrm{e}^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2\left(\frac{\partial}{\partial t}+\nu-\mu\right)\left(\frac{\partial}{\partial t}+\eta-\mu\right)C_1\\
&+\mathrm{e}^{-\nu t}\left(\frac{\partial}{\partial t}+\lambda-\nu\right)^2\left(\frac{\partial}{\partial t}+\mu-\nu\right)\left(\frac{\partial}{\partial t}+\eta-\nu\right)D_1\\
&+\mathrm{e}^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^2\left(\frac{\partial}{\partial t}+\mu-\eta\right)\left(\frac{\partial}{\partial t}+\nu-\eta\right)H_1\\
&=-\sum_{i,j,k,l,m=0}^{\infty} F_{0,m}(a,b,c,d,h)\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t}
\end{aligned} \tag{11}$$

Solution of Equation (10) is

$$B_1 = -\sum_{i,j,k,l,m=0}^{\infty} \frac{F_{1,m}(a,b,c,d,h)\,\mathrm{e}^{-\{(i-1)\lambda+j\mu+k\nu+l\eta\}t}}{\Omega_0\Omega_1\Omega_2\Omega_3} \tag{12}$$

where

$$\Omega_0 = \{(i-1)\lambda + j\mu + k\nu + l\eta\}$$

$$\Omega_1 = \{i\lambda + (j-1)\mu + k\nu + l\eta\}$$

$$\Omega_2 = \{i\lambda + j\mu + (k-1)\nu + l\eta\}$$

$$\Omega_3 = \{i\lambda + j\mu + k\nu + (l-1)\eta\}$$

Substituting the value of B_1 from Equation (12) into Equation (11), we obtain

$$\begin{aligned}
&\mathrm{e}^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\nu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\frac{\partial A_1}{\partial t}+\mathrm{e}^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2\left(\frac{\partial}{\partial t}+\nu-\mu\right)\left(\frac{\partial}{\partial t}+\eta-\mu\right)C_1\\
&+\mathrm{e}^{-\nu t}\left(\frac{\partial}{\partial t}+\lambda-\nu\right)^2\left(\frac{\partial}{\partial t}+\mu-\nu\right)\left(\frac{\partial}{\partial t}+\eta-\nu\right)D_1+\mathrm{e}^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^2\left(\frac{\partial}{\partial t}+\mu-\eta\right)\left(\frac{\partial}{\partial t}+\nu-\eta\right)H_1\\
&=-\sum_{i,j,k,l,m=0}^{\infty}F_{0,m}(a,b,c,d,h)\,\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t}-2\sum_{i,j,k,l,m=0}^{\infty}\frac{F_{1,m}(a,b,c,d,h)\,\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t}}{\Omega_0}
\end{aligned} \tag{13}$$

Different authors imposed different conditions according to the behavior of the systems, such as Alam ([20], [21]) imposed the condition $i_1\lambda_1 + i_2\lambda_2 + \cdots + i_n\lambda_n \le (i_1 + i_2 + \cdots + i_n)(\lambda_1 + \lambda_2 + \cdots + \lambda_n)/n$. Consequently, we have investigated the solutions for the case $\lambda \ll \mu \ll \nu \ll \eta$. Thus, we shall be able to separate the Equation (13) for the unknown functions A_1, C_1, D_1 and H_1; and solving them. Thus, substituting the values of A_1, B_1, C_1, D_1 and H_1 into the Equation (4) and integrating, we shall obtain the values of a, b, c, d and h. Equation (9) is a fifth order inhomogeneous linear differential equation. Therefore, it can be solved for u_1 by the well-known operator method. Hence, the determination of the first order approximate solution is completed.

3. Example

As an example of the above method, we consider the weakly nonlinear differential system

$$x^{(v)} + k_1 x^{(iv)} + k_2\dddot{x} + k_3\ddot{x} + k_4\dot{x} + k_5 x = -\varepsilon x^3 \tag{14}$$

Comparing (14) and (1), we obtain

$$\begin{aligned}
f^{(0)} &= \left(a^3+3a^2bt+3ab^2t^2+b^3t^3\right)\mathrm{e}^{-3\lambda t}+3\left(a^2+2abt+b^2t^2\right)c\mathrm{e}^{-(2\lambda+\mu)t}+3\left(a^2+2abt+b^2t^2\right)d\mathrm{e}^{-(2\lambda+\nu)t}\\
&+3\left(a^2+2abt+b^2t^2\right)h\mathrm{e}^{-(2\lambda+\eta)t}+3(a+bt)c^2\mathrm{e}^{-(\lambda+2\mu)t}+3(a+bt)d^2\mathrm{e}^{-(\lambda+2\nu)t}\\
&+3(a+bt)h^2\mathrm{e}^{-(\lambda+2\eta)t}+c^3\mathrm{e}^{-3\mu t}+3c^2d\mathrm{e}^{-(2\mu+\nu)t}+3cd^2\mathrm{e}^{-(\mu+2\nu)t}+d^3\mathrm{e}^{-3\nu t}\\
&+3\left(c^2\mathrm{e}^{-2\mu t}+2cd\mathrm{e}^{-(\mu+\nu)t}+d^2\mathrm{e}^{-2\nu t}\right)h\mathrm{e}^{-\eta t}+3ch^2\mathrm{e}^{-(2\eta+\mu)t}+3dh^2\mathrm{e}^{-(2\eta+\nu)t}\\
&+h^3\mathrm{e}^{-3\eta t}+6(a+bt)cd\mathrm{e}^{-(\lambda+\mu+\nu)t}+6(a+bt)ch\mathrm{e}^{-(\lambda+\mu+\eta)t}+6(a+bt)dh\mathrm{e}^{-(\lambda+\nu+\eta)t}
\end{aligned} \tag{15}$$

Now, comparing Equations (6) and (15), we obtain

$$\begin{aligned}
&\sum_{i,j,k,l,m=0}^{\infty}F_{0,m}(a,b,c,d,h)\,\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t}\\
&=a^3\mathrm{e}^{-3\lambda t}+3a^2c\mathrm{e}^{-(2\lambda+\mu)t}+3a^2d\mathrm{e}^{-(2\lambda+\nu)t}+3a^2h\mathrm{e}^{-(2\lambda+\eta)t}+3ac^2\mathrm{e}^{-(\lambda+2\mu)t}\\
&\quad+3ad^2\mathrm{e}^{-(\lambda+2\nu)t}+3ah^2\mathrm{e}^{-(\lambda+2\eta)t}+c^3\mathrm{e}^{-3\mu t}+3c^2d\mathrm{e}^{-(2\mu+\nu)t}+3cd^2\mathrm{e}^{-(\mu+2\nu)t}\\
&\quad+d^3\mathrm{e}^{-3\nu t}+3\left(c^2\mathrm{e}^{-2\mu t}+2cd\mathrm{e}^{-(\mu+\nu)t}+d^2\mathrm{e}^{-2\nu t}\right)h\mathrm{e}^{-\eta t}+3ch^2\mathrm{e}^{-(2\eta+\mu)t}\\
&\quad+3dh^2\mathrm{e}^{-(2\eta+\nu)t}+h^3\mathrm{e}^{-3\eta t}+6acd\mathrm{e}^{-(\lambda+\mu+\nu)t}+6ach\mathrm{e}^{-(\lambda+\mu+\eta)t}+6adh\mathrm{e}^{-(\lambda+\nu+\eta)t}
\end{aligned}$$

$$\sum_{i,j,k,l,m=0}^{\infty} F_{1,m}(a,b,c,d,h)\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t}$$
$$= 3a^2b\mathrm{e}^{-3\lambda t} + 6abc\mathrm{e}^{-(2\lambda+\mu)t} + 6abd\mathrm{e}^{-(2\lambda+\nu)t} + 6abh\mathrm{e}^{-(2\lambda+\eta)t} + 3bc^2\mathrm{e}^{-(\lambda+2\mu)t}$$
$$+ 3bd^2\mathrm{e}^{-(\lambda+2\nu)t} + 3bh^2\mathrm{e}^{-(\lambda+2\eta)t} + 6bcd\mathrm{e}^{-(\lambda+\mu+\nu)t} + 6bch\mathrm{e}^{-(\lambda+\mu+\eta)t} + 6bdh\mathrm{e}^{-(\lambda+\nu+\eta)t} \tag{16}$$

$$\sum_{i,j,k,l,m=0}^{\infty} F_{2,m}(a,b,c,d,h)\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t}$$
$$= 3ab^2\mathrm{e}^{-3\lambda t} + 3b^2c\mathrm{e}^{-(2\lambda+\mu)t} + 3b^2d\mathrm{e}^{-(2\lambda+\nu)t} + 3b^2h\mathrm{e}^{-(2\lambda+\eta)t}$$

$$\sum_{i,j,k,l,m=0}^{\infty} F_{3,m}(a,b,c,d,h)\mathrm{e}^{-(i\lambda+j\mu+k\nu+l\eta)t} = b^3\mathrm{e}^{-3\lambda t}$$

For Equation (14), the Equations (9) to (11) respectively become

$$\left(\frac{\partial}{\partial t}+\lambda\right)^2\left(\frac{\partial}{\partial t}+\mu\right)\left(\frac{\partial}{\partial t}+\nu\right)\left(\frac{\partial}{\partial t}+\eta\right)u_1$$
$$= -\left\{3ab^2t^2\mathrm{e}^{-3\lambda t} + 3b^2t^2c\mathrm{e}^{-(2\lambda+\mu)t} + 3b^2t^2d\mathrm{e}^{-(2\lambda+\nu)t} + 3b^2t^2h\mathrm{e}^{-(2\lambda+\eta)t} + b^3t^3\mathrm{e}^{-3\lambda t}\right\} \tag{17}$$

$$\mathrm{e}^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\nu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\frac{\partial B_1}{\partial t}$$
$$= -\left\{3a^2bt\mathrm{e}^{-3\lambda t} + 6abtc\mathrm{e}^{-(2\lambda+\mu)t} + 6abtd\mathrm{e}^{-(2\lambda+\nu)t} + 6abth\mathrm{e}^{-(2\lambda+\eta)t} + 3btc^2\mathrm{e}^{-(\lambda+2\mu)t}\right.$$
$$\left. + 3btd^2\mathrm{e}^{-(\lambda+2\nu)t} + 3bth^2\mathrm{e}^{-(\lambda+2\eta)t} + 6btcd\mathrm{e}^{-(\lambda+\mu+\nu)t} + 6btch\mathrm{e}^{-(\lambda+\mu+\eta)t} + 6btdh\mathrm{e}^{-(\lambda+\nu+\eta)t}\right\} \tag{18}$$

$$\mathrm{e}^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\nu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial A_1}{\partial t}+2B_1\right) + \mathrm{e}^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2\left(\frac{\partial}{\partial t}+\nu-\mu\right)\left(\frac{\partial}{\partial t}+\eta-\mu\right)C_1$$
$$+ \mathrm{e}^{-\nu t}\left(\frac{\partial}{\partial t}+\lambda-\nu\right)^2\left(\frac{\partial}{\partial t}+\mu-\nu\right)\left(\frac{\partial}{\partial t}+\eta-\nu\right)D_1 + \mathrm{e}^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^2\left(\frac{\partial}{\partial t}+\mu-\eta\right)\left(\frac{\partial}{\partial t}+\nu-\eta\right)H_1$$
$$= -\left\{a^3\mathrm{e}^{-3\lambda t} + 3a^2c\mathrm{e}^{-(2\lambda+\mu)t} + 3a^2d\mathrm{e}^{-(2\lambda+\nu)t} + 3a^2h\mathrm{e}^{-(2\lambda+\eta)t} + 3ac^2\mathrm{e}^{-(\lambda+2\mu)t} + 3ad^2\mathrm{e}^{-(\lambda+2\nu)t} + 3ah^2\mathrm{e}^{-(\lambda+2\eta)t}\right.$$
$$+ c^3\mathrm{e}^{-3\mu t} + 3c^2d\mathrm{e}^{-(2\mu+\nu)t} + 3cd^2\mathrm{e}^{-(\mu+2\nu)t} + d^3\mathrm{e}^{-3\nu t} + 3\left(c^2\mathrm{e}^{-2\mu t} + 2cd\mathrm{e}^{-(\mu+\nu)t} + d^2\mathrm{e}^{-2\nu t}\right)h\mathrm{e}^{-\eta t}$$
$$\left. + 3ch^2\mathrm{e}^{-(2\eta+\mu)t} + 3dh^2\mathrm{e}^{-(2\eta+\nu)t} + h^3\mathrm{e}^{-3\eta t} + 6acd\mathrm{e}^{-(\lambda+\mu+\nu)t} + 6ach\mathrm{e}^{-(\lambda+\mu+\eta)t} + 6adh\mathrm{e}^{-(\lambda+\nu+\eta)t}\right\} \tag{19}$$

The solution of the Equation (18) is

$$B_1 = l_1a^2b\mathrm{e}^{-2\lambda t} + l_2abc\mathrm{e}^{-(\lambda+\mu)t} + l_3bc^2\mathrm{e}^{-2\mu t} + l_4abd\mathrm{e}^{-(\lambda+\nu)t} + l_5bcd\mathrm{e}^{-(\mu+\nu)t}$$
$$+ l_6abh\mathrm{e}^{-(\lambda+\eta)t} + l_7bch\mathrm{e}^{-(\mu+\eta)t} + l_8bd^2\mathrm{e}^{-2\nu t} + l_9bdh\mathrm{e}^{-(\nu+\eta)t} + l_{10}bh^2\mathrm{e}^{-2\eta t} \tag{20}$$

where $l_1 = \dfrac{3}{2\lambda(\mu-3\lambda)(\nu-3\lambda)(\eta-3\lambda)}$, $l_2 = \dfrac{-3}{\lambda(\lambda+\mu)(\nu-\mu-2\lambda)(\eta-2\lambda-\mu)}$,

$$l_3 = \frac{-3}{2\mu(\lambda+\mu)(\nu-2\mu-\lambda)(\eta-2\mu-\lambda)},\quad l_4 = \frac{-3}{\lambda(\lambda+\nu)(\mu-\nu-2\lambda)(\eta-\nu-2\lambda)},$$

$$l_5 = \frac{6}{(\lambda+\nu)(\mu+\nu)(\lambda+\mu)(\eta-\lambda-\mu-\nu)},\quad l_6 = \frac{-3}{\lambda(\lambda+\eta)(\mu-2\lambda-\eta)(\nu-2\lambda-\eta)},$$

$$l_7 = \frac{6}{(\lambda+\eta)(\mu+\eta)(\lambda+\mu)(\nu-\lambda-\mu-\eta)},\quad l_8 = \frac{-3}{2\nu(\lambda+\nu)(\mu-\lambda-2\nu)(\eta-2\nu-\lambda)},$$

$$l_9=\frac{6}{(\lambda+\eta)(\lambda+\nu)(\nu+\eta)(\mu-\lambda-\nu-\eta)},\quad l_{10}=\frac{-3}{2\eta(\mu-\lambda-2\eta)(\nu-\lambda-2\eta)(\eta+\lambda)}$$

Putting the value of B_1 from Equation (21) into Equation (20), we obtain

$$\begin{aligned}
&e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\nu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\left(\frac{\partial A_1}{\partial t}\right)+e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2\left(\frac{\partial}{\partial t}+\nu-\mu\right)\left(\frac{\partial}{\partial t}+\eta-\mu\right)C_1\\
&+e^{-\nu t}\left(\frac{\partial}{\partial t}+\lambda-\nu\right)^2\left(\frac{\partial}{\partial t}+\mu-\nu\right)\left(\frac{\partial}{\partial t}+\eta-\nu\right)D_1+e^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^2\left(\frac{\partial}{\partial t}+\mu-\eta\right)\left(\frac{\partial}{\partial t}+\nu-\eta\right)H_1\\
&=-\Big\{a^3e^{-3\lambda t}+3a^2ce^{-(2\lambda+\mu)t}+3a^2de^{-(2\lambda+\nu)t}+3a^2he^{-(2\lambda+\eta)t}+3ac^2e^{-(\lambda+2\mu)t}+3ad^2e^{-(\lambda+2\nu)t}+3ah^2e^{-(\lambda+2\eta)t}\\
&\quad+c^3e^{-3\mu t}+3c^2de^{-(2\mu+\nu)t}+3cd^2e^{-(\mu+2\nu)t}+d^3e^{-3\nu t}+3\left(c^2e^{-2\mu t}+2cde^{-(\mu+\nu)t}+d^2e^{-2\nu t}\right)he^{-\eta t}\\
&\quad+3ch^2e^{-(2\eta+\mu)t}+3dh^2e^{-(2\eta+\nu)t}+h^3e^{-3\eta t}+6acde^{-(\lambda+\mu+\nu)t}+6ache^{-(\lambda+\mu+\eta)t}+6adhe^{-(\lambda+\nu+\eta)t}\Big\}\\
&\quad-2l_1a^2be^{-3\lambda t}(\mu-3\lambda)(\nu-3\lambda)(\eta-3\lambda)+4\lambda l_2abce^{-(2\lambda+\mu)t}(\nu-2\lambda-\mu)(\eta-2\lambda-\mu)\\
&\quad+2l_3bc^2e^{-(2\mu+\lambda)t}(\lambda+\mu)(\nu-2\mu-\lambda)(\eta-2\mu-\lambda)+4\lambda l_4abde^{-(2\lambda+\nu)t}(\mu-2\lambda-\nu)(\eta-2\lambda-\nu)\\
&\quad-2l_5bcde^{-(\lambda+\mu+\nu)t}(\nu+\lambda)(\lambda+\mu)(\eta-\nu-\lambda-\mu)+4\lambda l_6abhe^{-(2\lambda+\eta)t}(\mu-2\lambda-\eta)(\nu-2\lambda-\eta)\\
&\quad-2l_7bche^{-(\lambda+\mu+\eta)t}(\lambda+\eta)(\nu-\lambda-\mu-\eta)(\lambda+\mu)+2l_8bd^2e^{-(2\nu+\lambda)t}(\mu-2\nu-\lambda)(\nu+\lambda)(\eta-2\nu-\lambda)\\
&\quad-2l_9bdhe^{-(\lambda+\nu+\eta)t}(\mu-\nu-\lambda-\eta)(\lambda+\eta)(\nu+\lambda)+2l_{10}bh^2e^{-(2\eta+\lambda)t}(\mu-2\eta-\lambda)(\nu-2\eta-\lambda)(\lambda+\eta)
\end{aligned}\tag{21}$$

Since the relation $\lambda\ll\mu\ll\nu\ll\eta$ among the eigenvalues, so the Equation (21) can be separated for the unknown functions A_1,C_1,D_1 and H_1 in the following way:

$$\begin{aligned}
&e^{-\lambda t}\left(\frac{\partial}{\partial t}+\mu-\lambda\right)\left(\frac{\partial}{\partial t}+\nu-\lambda\right)\left(\frac{\partial}{\partial t}+\eta-\lambda\right)\frac{\partial A}{\partial t}=-\left(a^3-2l_1a^2b\right)(\mu-3\lambda)(\nu-3\lambda)(\eta-3\lambda)e^{-3\lambda t}\\
&e^{-\mu t}\left(\frac{\partial}{\partial t}+\lambda-\mu\right)^2\left(\frac{\partial}{\partial t}+\nu-\mu\right)\left(\frac{\partial}{\partial t}+\eta-\mu\right)C_1\\
&=4\lambda l_2abce^{-(2\lambda+\mu)t}(\nu-2\lambda-\mu)(\eta-2\lambda-\mu)-3a^2ce^{-(2\lambda+\mu)t}\\
&\quad+2l_3bc^2e^{-(2\mu+\lambda)t}(\lambda+\mu)(\nu-2\mu-\lambda)(\eta-2\mu-\lambda)-3ac^2e^{-(\lambda+2\mu)t}-c^3e^{-3\mu t}\\
&e^{-\nu t}\left(\frac{\partial}{\partial t}+\lambda-\nu\right)^2\left(\frac{\partial}{\partial t}+\mu-\nu\right)\left(\frac{\partial}{\partial t}+\eta-\nu\right)D_1\\
&=-3a^2de^{-(2\lambda+\nu)t}-3ad^2e^{-(\lambda+2\nu)t}-3c^2de^{-(2\mu+\nu)t}-3cd^2e^{-(\mu+2\nu)t}-d^3e^{-3\nu t}-6acde^{-(\lambda+\mu+\nu)t}\\
&\quad+4\lambda l_4abde^{-(2\lambda+\nu)t}(\mu-2\lambda-\nu)(\eta-2\lambda-\nu)-2l_5bcde^{-(\lambda+\mu+\nu)t}(\nu+\lambda)(\lambda+\mu)(\eta-\nu-\lambda-\mu)\\
&\quad+2l_8bd^2e^{-(2\nu+\lambda)t}(\mu-2\nu-\lambda)(\eta-2\nu-\lambda)(\nu+\lambda)\\
&e^{-\eta t}\left(\frac{\partial}{\partial t}+\lambda-\eta\right)^2\left(\frac{\partial}{\partial t}+\mu-\eta\right)\left(\frac{\partial}{\partial t}+\nu-\eta\right)H_1\\
&=-3a^2he^{-(2\lambda+\eta)t}-3ah^2e^{-(\lambda+2\eta)t}-3(c^2e^{-2\mu t}+2cde^{-(\mu+\nu)t}+d^2e^{-2\nu t})he^{-\eta t}-3ch^2e^{-(2\eta+\mu)t}\\
&\quad-3dh^2e^{-(2\eta+\nu)t}-h^3e^{-3\eta t}-6ache^{-(\lambda+\mu+\eta)t}-6adhe^{-(\lambda+\nu+\eta)t}+4\lambda l_6abhe^{-(2\lambda+\eta)t}(\mu-2\lambda-\eta)(\nu-2\lambda-\eta)\\
&\quad-2l_7bche^{-(\lambda+\mu+\eta)t}(\lambda+\eta)(\lambda+\mu)(\nu-\lambda-\mu-\eta)-2l_9bdhe^{-(\lambda+\nu+\eta)t}(\mu-\nu-\lambda-\eta)(\lambda+\eta)(\nu+\lambda)\\
&\quad+2l_{10}bh^2e^{-(2\eta+\lambda)t}(\mu-2\eta-\lambda)(\nu-2\eta-\lambda)(\lambda+\eta)
\end{aligned}\tag{22}$$

Solving Equation (22), we obtain

$$A_1=m_1a^2be^{-2\lambda t}+m_2a^3e^{-2\lambda t}\tag{23}$$

where $$m_1 = \frac{l_1}{\lambda},\quad m_2 = \frac{1}{2\lambda(\mu-3\lambda)(\nu-3\lambda)(\eta-3\lambda)}$$

$$C_1 = k_1abc\mathrm{e}^{-2\lambda t} + k_2bc^2\mathrm{e}^{-(\lambda+\mu)t} + k_3a^2c\mathrm{e}^{-2\lambda t} + k_4ac^2\mathrm{e}^{-(\lambda+\mu)t} + k_5c^3\mathrm{e}^{-2\mu t} \tag{24}$$

where $$k_1 = \frac{4l_2\lambda}{(\lambda+\mu)^2},\quad k_2 = \frac{l_3(\lambda+\mu)}{2\mu^2},\quad k_3 = -\frac{3}{(\lambda+\mu)^2(\nu-2\lambda-\mu)(\eta-2\lambda-\mu)},$$

$$k_4 = -\frac{3}{4\mu^2(\nu-2\mu-\lambda)(\eta-2\mu-\lambda)},\quad k_5 = -\frac{1}{(\lambda-3\mu)^2(\nu-3\mu)(\eta-3\mu)}$$

$$\begin{aligned} D_1 = {} & p_1abd\mathrm{e}^{-2\lambda t} + p_2bcd\mathrm{e}^{-(\lambda+\mu)t} + p_3bd^2\mathrm{e}^{-(\lambda+\nu)t} + p_4a^2d\mathrm{e}^{-2\lambda t} + p_5acd\mathrm{e}^{-(\lambda+\mu)t} \\ & + p_6dc^2\mathrm{e}^{-2\mu t} + p_7ad^2\mathrm{e}^{-(\lambda+\nu)t} + p_8cd^2\mathrm{e}^{-(\mu+\nu)t} + p_9d^3\mathrm{e}^{-2\nu t} \end{aligned} \tag{25}$$

where $$p_1 = \frac{4l_4\lambda}{(\lambda+\nu)^2},\quad p_2 = \frac{2l_5(\lambda+\mu)}{(\nu+\mu)^2},\quad p_3 = \frac{l_8(\lambda+\nu)}{2\nu^2},\quad p_4 = \frac{-3}{(\lambda+\nu)^2(\mu-2\lambda-\nu)(\eta-2\lambda-\nu)},$$

$$p_5 = \frac{-6}{(\mu+\nu)^2(\lambda+\nu)(\eta-\lambda-\mu-\nu)},\quad p_6 = \frac{3}{(\lambda-2\mu-\nu)^2(\mu+\nu)(\eta-2\mu-\nu)},\quad p_7 = \frac{-3}{4\nu^2(\mu-2\nu-\lambda)},$$

$$p_8 = \frac{3}{2\nu(\lambda-2\nu-\mu)^2(\eta-2\nu-\mu)},\quad p_9 = \frac{-1}{(\lambda-3\nu)^2(\mu-3\nu)(\eta-3\nu)}$$

$$\begin{aligned} H_1 = {} & q_1abh\mathrm{e}^{-2\lambda t} + q_2bch\mathrm{e}^{-(\lambda+\mu)t} + q_3bdh\mathrm{e}^{-(\lambda+\nu)t} + q_4bh^2\mathrm{e}^{-(\lambda+\eta)t} + q_5a^2h\mathrm{e}^{-2\lambda t} \\ & + q_6ach\mathrm{e}^{-(\lambda+\mu)t} + q_7c^2h\mathrm{e}^{-2\mu t} + q_8adh\mathrm{e}^{-(\lambda+\nu)t} + q_9ah^2\mathrm{e}^{-(\lambda+\eta)t} + q_{10}cdh\mathrm{e}^{-(\mu+\nu)t} \\ & + q_{11}ch^2\mathrm{e}^{-(\mu+\eta)t} + q_{12}d^2h\mathrm{e}^{-2\nu t} + q_{13}dh^2\mathrm{e}^{-(\nu+\eta)t} + q_{14}h^3\mathrm{e}^{-2\eta t} \end{aligned} \tag{26}$$

where $$q_1 = \frac{4l_6\lambda}{(\lambda+\eta)^2},\quad q_2 = \frac{2l_7(\lambda+\eta)}{(\mu+\eta)^2},\quad q_3 = \frac{2l_9(\lambda+\nu)}{(\nu+\eta)^2},\quad q_4 = \frac{l_{10}(\lambda+\eta)}{2\eta^2},$$

$$q_5 = \frac{-3}{(\lambda+\eta)^2(\mu-2\lambda-\eta)(\nu-2\lambda-\eta)},\quad q_6 = \frac{6}{(\mu+\eta)^2(\lambda+\eta)(\nu-\lambda-\mu-\eta)},$$

$$q_7 = \frac{3}{(\lambda-2\mu-\eta)^2(\mu+\eta)(\nu-2\mu-\eta)},\quad q_8 = \frac{6}{(\nu+\eta)^2(\lambda+\eta)(\mu-\nu-\lambda-\eta)},$$

$$q_9 = \frac{-3}{4\eta^2(\mu-2\eta-\lambda)(\nu-2\eta-\lambda)},\quad q_{10} = \frac{-6}{(\lambda-\mu-\nu-\eta)^2(\nu+\eta)(\mu+\eta)},$$

$$q_{11} = \frac{3}{2\eta(\lambda-2\eta-\mu)^2(\nu-2\eta-\mu)},\quad q_{12} = \frac{3}{(\lambda-2\nu-\eta)^2(\mu-2\nu-\eta)(\nu+\eta)},$$

$$q_{13} = \frac{3}{2\eta(\lambda-2\eta-\nu)^2(\mu-2\eta-\nu)},\quad q_{14} = \frac{-1}{(\lambda-3\eta)^2(\mu-3\eta)(\nu-3\eta)}$$

And the solution of the Equation (18) is

$$\begin{aligned} u_1 = {} & ab^2(n_1t^2+n_2t+n_3)\mathrm{e}^{-3\lambda t} + b^3(n_4t^3+n_5t^2+n_6t+n_7)\mathrm{e}^{-3\lambda t} + b^2c(n_8t^2+n_9t+n_{10})\mathrm{e}^{-(2\lambda+\mu)t} \\ & + b^2d(n_{11}t^2+n_{12}t+n_{13})\mathrm{e}^{-(2\lambda+\nu)t} + b^2h(n_{14}t^2+n_{15}t+n_{16})\mathrm{e}^{-(2\lambda+\eta)t} \end{aligned} \tag{27}$$

where $$n_1 = \frac{3}{4\lambda^2(3\lambda-\mu)(3\lambda-\nu)(3\lambda-\eta)},\quad n_2 = 2n_1\left[\frac{1}{(3\lambda-\mu)} + \frac{1}{(3\lambda-\nu)} + \frac{1}{(3\lambda-\eta)} + \frac{1}{\lambda}\right],$$

$$n_3 = 2n_1\left[\frac{1}{(3\lambda-\eta)^2}+\frac{1}{(3\lambda-\eta)(3\lambda-\nu)}+\frac{1}{(3\lambda-\nu)^2}+\left(\frac{1}{\lambda}+\frac{1}{3\lambda-\mu}\right)\left(\frac{1}{3\lambda-\eta}+\frac{1}{3\lambda-\nu}\right)\right.$$
$$\left.+\frac{3}{4\lambda^2}+\frac{1}{\lambda(3\lambda-\mu)}+\frac{1}{(3\lambda-\mu)^2}\right],$$

$$n_4 = \frac{1}{4\lambda^2(3\lambda-\mu)(3\lambda-\nu)(3\lambda-\eta)},\quad n_5 = 3n_4\left[\frac{1}{(3\lambda-\mu)}+\frac{1}{(3\lambda-\nu)}+\frac{1}{(3\lambda-\eta)}+\frac{1}{\lambda}\right],$$

$$n_6 = 6n_4\left[\frac{1}{(3\lambda-\eta)^2}+\frac{1}{(3\lambda-\eta)(3\lambda-\nu)}+\frac{1}{(3\lambda-\nu)^2}+\left(\frac{1}{\lambda}+\frac{1}{3\lambda-\mu}\right)\left(\frac{1}{3\lambda-\eta}+\frac{1}{3\lambda-\nu}\right)\right.$$
$$\left.+\frac{3}{4\lambda^2}+\frac{1}{\lambda(3\lambda-\mu)}+\frac{1}{(3\lambda-\mu)^2}\right],$$

$$n_7 = 6n_4\left[\frac{1}{(3\lambda-\eta)^3}+\frac{1}{(3\lambda-\eta)^2(3\lambda-\nu)}+\frac{1}{(3\lambda-\nu)^2(3\lambda-\eta)}+\frac{1}{(3\lambda-\nu)^3}\right.$$
$$+\left(\frac{1}{\lambda}+\frac{1}{3\lambda-\mu}\right)\left(\frac{1}{(3\lambda-\eta)^2}+\frac{1}{(3\lambda-\nu)(3\lambda-\eta)}+\frac{1}{(3\lambda-\nu)^2}\right)$$
$$+\left(\frac{1}{3\lambda-\eta}+\frac{1}{(3\lambda-\nu)}\right)\left(\frac{3}{4\lambda^2}+\frac{1}{\lambda(3\lambda-\mu)}+\frac{1}{(3\lambda-\mu)^2}\right)$$
$$\left.+\frac{1}{2\lambda^3}+\frac{3}{4\lambda^2(3\lambda-\mu)}+\frac{1}{\lambda(3\lambda-\mu)^2}+\frac{1}{(3\lambda-\mu)^3}\right],$$

$$n_8 = \frac{3}{2\lambda(\lambda+\mu)^2(2\lambda+\mu-\nu)(2\lambda+\mu-\eta)},\quad n_9 = 2n_8\left[\frac{1}{2\lambda+\mu-\nu}+\frac{1}{2\lambda+\mu-\eta}+\frac{1}{2\lambda}+\frac{2}{\lambda+\mu}\right],$$

$$n_{10} = 2n_8\left[\frac{1}{(2\lambda+\mu-\eta)^2}+\frac{1}{(2\lambda+\mu-\eta)(2\lambda+\mu-\nu)}+\frac{1}{(2\lambda+\mu-\nu)^2}\right.$$
$$\left.+\left(\frac{1}{2\lambda}+\frac{2}{\lambda+\mu}\right)\left(\frac{1}{2\lambda+\mu-\eta}+\frac{1}{2\lambda+\mu-\nu}\right)+\frac{1}{4\lambda^2}+\frac{1}{\lambda(\lambda+\mu)}+\frac{3}{(\lambda+\mu)^2}\right],$$

$$n_{11} = \frac{3}{2\lambda(\lambda+\nu)^2(2\lambda+\nu-\mu)(2\lambda+\nu-\eta)},\quad n_{12} = 2n_{11}\left[\frac{1}{2\lambda+\nu-\mu}+\frac{1}{2\lambda+\nu-\eta}+\frac{1}{2\lambda}+\frac{2}{\lambda+\nu}\right],$$

$$n_{13} = 2n_{11}\left[\frac{1}{(2\lambda+\nu-\eta)^2}+\frac{1}{(2\lambda+\nu-\eta)(2\lambda+\nu-\mu)}+\frac{1}{(2\lambda+\nu-\mu)^2}\right.$$
$$\left.+\left(\frac{1}{2\lambda}+\frac{2}{\lambda+\nu}\right)\left(\frac{1}{2\lambda+\nu-\eta}+\frac{1}{2\lambda+\nu-\mu}\right)+\frac{1}{4\lambda^2}+\frac{1}{\lambda(\lambda+\nu)}+\frac{3}{(\lambda+\nu)^2}\right],$$

$$n_{14} = \frac{3}{2\lambda(\lambda+\eta)^2(2\lambda+\eta-\mu)(2\lambda+\eta-\nu)},\quad n_{15} = 2n_{14}\left[\frac{1}{2\lambda+\eta-\mu}+\frac{1}{2\lambda+\eta-\nu}+\frac{1}{2\lambda}+\frac{2}{\lambda+\eta}\right],$$

$$n_{16}=2n_{14}\left[\frac{1}{(2\lambda+\eta-\nu)^2}+\frac{1}{(2\lambda+\eta-\mu)(2\lambda+\eta-\nu)}+\frac{1}{(2\lambda+\eta-\mu)^2}\right.$$
$$\left.+\left(\frac{1}{2\lambda}+\frac{2}{\lambda+\eta}\right)\left(\frac{1}{2\lambda+\eta-\mu}+\frac{1}{2\lambda+\eta-\nu}\right)+\frac{1}{4\lambda^2}+\frac{1}{\lambda(\lambda+\eta)}+\frac{3}{(\lambda+\eta)^2}\right]$$

Substituting the values of A_1, B_1, C_1, D_1 and H_1 from Equations (23), (20), (24), (25) and (26) into Equation (4), we obtain

$$\dot{a}=\varepsilon\left[m_1a^2b\mathrm{e}^{-2\lambda t}+m_2a^3\mathrm{e}^{-2\lambda t}\right]$$
$$\dot{b}=\varepsilon\left[l_1a^2b\mathrm{e}^{-2\lambda t}+l_2abc\mathrm{e}^{-(\lambda+\mu)t}+l_3bc^2\mathrm{e}^{-2\mu t}+l_4abd\mathrm{e}^{-(\lambda+\nu)t}+l_5bcd\mathrm{e}^{-(\mu+\nu)t}\right.$$
$$\left.+l_6abh\mathrm{e}^{-(\lambda+\eta)t}+l_7bch\mathrm{e}^{-(\mu+\eta)t}+l_8bd^2\mathrm{e}^{-2\nu t}+l_9bdh\mathrm{e}^{-(\nu+\eta)t}+l_{10}bh^2\mathrm{e}^{-2\eta t}\right]$$
$$\dot{c}=\varepsilon\left[k_1abc\mathrm{e}^{-2\lambda t}+k_2bc^2\mathrm{e}^{-(\lambda+\mu)t}+k_3a^2c\mathrm{e}^{-2\lambda t}+k_4ac^2\mathrm{e}^{-(\lambda+\mu)t}+k_5c^3\mathrm{e}^{-2\mu t}\right] \tag{28}$$
$$\dot{d}=\varepsilon\left[p_1abd\mathrm{e}^{-2\lambda t}+p_2bcd\mathrm{e}^{-(\lambda+\mu)t}+p_3bd^2\mathrm{e}^{-(\lambda+\nu)t}+p_4a^2d\mathrm{e}^{-2\lambda t}+p_5acd\mathrm{e}^{-(\lambda+\mu)t}\right.$$
$$\left.+p_6dc^2\mathrm{e}^{-2\mu t}+p_7ad^2\mathrm{e}^{-(\lambda+\nu)t}+p_8cd^2\mathrm{e}^{-(\mu+\nu)t}+p_9d^3\mathrm{e}^{-2\nu t}\right]$$
$$\dot{h}=\varepsilon\left[q_1abh^{-2\lambda t}+q_2bch\mathrm{e}^{-(\lambda+\mu)t}+q_3bdh\mathrm{e}^{-(\lambda+\nu)t}+q_4bh^2\mathrm{e}^{-(\lambda+\eta)t}+q_5a^2h\mathrm{e}^{-2\lambda t}\right.$$
$$+q_6ach\mathrm{e}^{-(\lambda+\mu)t}+q_7c^2h\mathrm{e}^{-2\mu t}+q_8adh\mathrm{e}^{-(\lambda+\nu)t}+q_9ah^2\mathrm{e}^{-(\lambda+\eta)t}+q_{10}cdh\mathrm{e}^{-(\mu+\nu)t}$$
$$\left.+q_{11}ch^2\mathrm{e}^{-(\mu+\eta)t}+q_{12}d^2h\mathrm{e}^{-2\nu t}+q_{13}dh^2\mathrm{e}^{-(\nu+\eta)t}+q_{14}h^3\mathrm{e}^{-2\eta t}\right]$$

Here the equations of (28) have no exact solutions, but since $\dot{a},\dot{b},\dot{c},\dot{d}$ and $\dot{h}$ are proportional to the small parameter ε, so they are slowly varying functions of time t. Therefore, it is possible to replace a,b,c,d and h by their respective values obtained in linear case in the right hand side of Equation (28). Murty and Deekshatulu [22] and Murty *et al.* [5] first made such type of amendment to solve similar type of nonlinear equations. Thus, the solution of (28) is

$$a=a_0+\varepsilon\left[m_1a_0^2b_0\frac{1-\mathrm{e}^{-2\lambda t}}{2\lambda}+m_2a_0^3\frac{1-\mathrm{e}^{-2\lambda t}}{2\lambda}\right]$$
$$b=b_0+\varepsilon\left[l_1a_0^2b_0\frac{1-\mathrm{e}^{-2\lambda t}}{2\lambda}+l_2a_0b_0c_0\frac{1-\mathrm{e}^{-(\lambda+\mu)t}}{\lambda+\mu}+l_3b_0c_0^2\frac{1-\mathrm{e}^{-2\mu t}}{2\mu}+l_4a_0b_0d_0\frac{1-\mathrm{e}^{-(\lambda+\nu)t}}{\lambda+\nu}\right.$$
$$+l_5b_0c_0d_0\frac{1-\mathrm{e}^{-(\mu+\nu)t}}{\mu+\nu}+l_6a_0b_0h_0\frac{1-\mathrm{e}^{-(\lambda+\eta)t}}{\lambda+\eta}+l_7b_0c_0h_0\frac{1-\mathrm{e}^{-(\mu+\eta)t}}{\mu+\eta}+l_8b_0d_0^2\frac{1-\mathrm{e}^{-2\nu t}}{2\nu} \tag{29}$$
$$\left.+l_9b_0d_0h_0\frac{1-\mathrm{e}^{-(\nu+\eta)t}}{\nu+\eta}+l_{10}b_0h_0^2\frac{1-\mathrm{e}^{-2\eta t}}{2\eta}\right]$$
$$c=c_0+\varepsilon\left[k_1a_0b_0c_0\frac{1-\mathrm{e}^{-2\lambda t}}{2\lambda}+k_2b_0c_0^2\frac{1-\mathrm{e}^{-(\lambda+\mu)t}}{\lambda+\mu}+k_3a_0^2c_0\frac{1-\mathrm{e}^{2\lambda t}}{2\lambda}+k_4a_0c_0^2\frac{1-\mathrm{e}^{-(\lambda+\mu)t}}{\lambda+\mu}+k_5c_0^3\frac{1-\mathrm{e}^{-2\mu t}}{2\mu}\right]$$
$$d=d_0+\varepsilon\left[p_1a_0b_0d_0\frac{1-e^{-2\lambda t}}{2\lambda}+p_2b_0c_0d_0\frac{1-\mathrm{e}^{-(\lambda+\mu)t}}{\lambda+\mu}+p_3b_0d_0^2\frac{1-\mathrm{e}^{-(\lambda+\nu)t}}{\lambda+\nu}+p_4a_0^2d_0\frac{1-\mathrm{e}^{-2\lambda t}}{2\lambda}\right.$$
$$\left.+p_5a_0c_0d_0\frac{1-\mathrm{e}^{-(\lambda+\mu)t}}{\lambda+\mu}+p_6d_0c_0^2\frac{1-\mathrm{e}^{-2\mu t}}{2\mu}+p_7a_0d_0^2\frac{1-\mathrm{e}^{-(\lambda+\nu)t}}{\lambda+\nu}+p_8c_0d_0^2\frac{1-\mathrm{e}^{-(\mu+\nu)t}}{\mu+\nu}+p_9d_0^3\frac{1-\mathrm{e}^{-2\nu t}}{2\nu}\right]$$

$$h = h_0 + \varepsilon \left[q_1 a_0 b_0 h_0 \frac{1-e^{-2\lambda t}}{2\lambda} + q_2 b_0 c_0 h_0 \frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + q_3 b_0 d_0 h_0 \frac{1-e^{-(\lambda+\nu)t}}{\lambda+\nu} + q_4 b_0 h_0^2 \frac{1-e^{-(\lambda+\eta)t}}{\lambda+\eta} \right.$$
$$+ q_5 a_0^2 h_0 \frac{1-e^{-2\lambda t}}{2\lambda} + q_6 a_0 c_0 h_0 \frac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + q_7 c_0^2 h_0 \frac{1-e^{-2\mu t}}{2\mu} + q_8 a_0 d_0 h_0 \frac{1-e^{-(\lambda+\nu)t}}{\lambda+\nu}$$
$$+ q_9 a_0 h_0^2 \frac{1-e^{-(\lambda+\eta)t}}{\lambda+\eta} + q_{10} c_0 d_0 h_0 \frac{1-e^{-(\mu+\nu)t}}{\mu+\nu} + q_{11} c_0 h_0^2 \frac{1-e^{-(\mu+\eta)t}}{\mu+\eta} + q_{12} d_0^2 h_0 \frac{1-e^{-2\nu t}}{2\nu}$$
$$\left. + q_{13} d_0 h_0^2 \frac{1-e^{-(\nu+\eta)t}}{\nu+\eta} + q_{14} h_0^3 \frac{1-e^{-2\eta t}}{2\eta} \right]$$

We, therefore, obtain the first approximate solution of the Equation (14) as

$$x(t,\varepsilon) = (a+bt)e^{-\lambda t} + ce^{-\mu t} + de^{-\nu t} + he^{-\eta t} + \varepsilon u_1 \tag{30}$$

where a,b,c,d and h are given by the Equation (29) and u_1 is given by (27).

4. Results and Discussion

Generally, the perturbation solution is compared to the numerical solution in order to test the accuracy of the approximate solution obtained by a certain perturbation method. First, we have considered the eigenvalues $\lambda = 0.15$, $\mu = 0.80$, $\nu = 3.82$ and $\eta = 14.5$. We have then computed $x(t,\varepsilon)$ using (30), in which a,b,c,d and h are obtained from (30) and u_1 is calculated from Equation (27) together with initial conditions $a_0 = 0.30$, $b_0 = 0.1$, $c_0 = 0.3$, $d_0 = 0.45$ and $h_0 = 0.4$ when $\varepsilon = 0.1$. Throughout the paper all the figures (**Figures 1-3**) represent the perturbation results which are displayed by the continuous line and the corresponding numerical results have been computed by fourth-order *Runge-Kutta* method, which are plotted by a discrete line.

Again, we have computed $x(t,\varepsilon)$ from (30) by considering $\lambda = 0.1$, $\mu = 0.75$, $\nu = 3.82$ and $\eta = 14.3$. We have computed $x(t,\varepsilon)$ using (30), in which a,b,c,d and h are obtained from (30) and u_1 is calculated from Equation (27) together with initial conditions $a_0 = 0.25$, $b_0 = 0.1$, $c_0 = 0.4$, $d_0 = 0.35$ and $h_0 = 0.5$ when $\varepsilon = 0.1$.

Finally, we have computed $x(t,\varepsilon)$ from (30) by considering $\lambda = 0.2$, $\mu = 0.85$, $\nu = 3.9$ and $\eta = 14.7$. We have computed $x(t,\varepsilon)$ using (30), in which a,b,c,d and h are obtained from (30) and u_1 is calculated from Equation (27) together with initial conditions $a_0 = 0.35$, $b_0 = 0.2$, $c_0 = 0.3$, $d_0 = 0.4$ and $h_0 = 0.45$ when $\varepsilon = 0.1$.

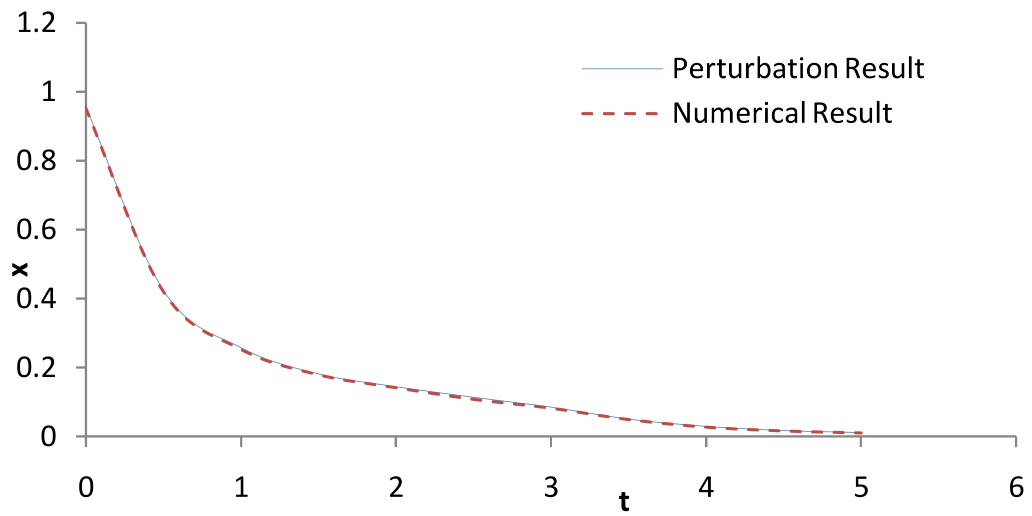

Figure 1. Comparison between perturbation and numerical results.

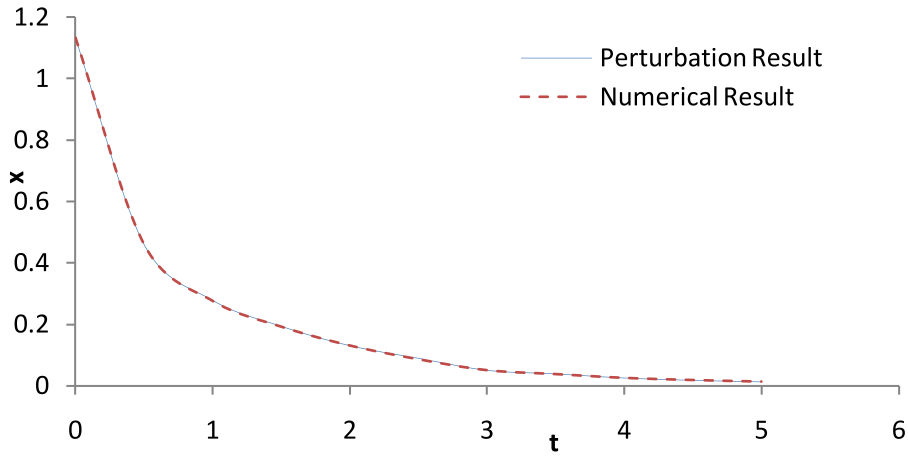

Figure 2. Comparison between perturbation and numerical results.

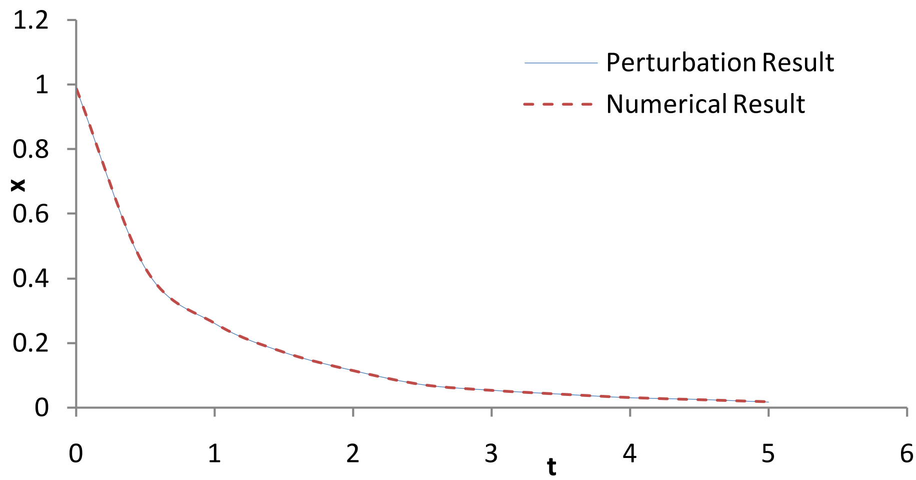

Figure 3. Comparison between perturbation and numerical results.

5. Conclusion

In this paper, we have obtained an analytical approximate solution based upon the KBM method of fifth order critically damped nonlinear systems. Moreover, we have shown in this paper that the results obtained by the proposed method correspond satisfactorily to the numerical results obtained by the fourth order *Runge-Kutta* method.

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