The Exponentiated Kumaraswamy-Exponential Distribution

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Abstract

A new continuous distribution called exponentiated Kumaraswamy-exponential that extends the exponential distribution and some other distributions is proposed and studied. Several structural properties of the new distribution were investigated, including the moments, hazard function, mean deviations and Rényi entropy. Moreover, we discuss the maximum likelihood estimation of this distribution. An application reveals that the model proposed can be very useful in fitting real data.

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1 INTRODUCTION

According to [1], the exponential distribution is perhaps the most widely applied statistical distribution for problems in reliability. For many years, researchers have been developing various extensions and modified forms of the exponential distribution. Recently, [2] introduced an extension of the exponential distribution called the transmuted exponentiated exponential distribution. [3] introduced and studied the gamma-exponentiated exponential distribution. [4] studied the exponentiated exponential distribution and discussed its various properties. [5] proposed the beta generalized exponential distribution and discussed its various properties. [6] introduced and studied a three-parameter distribution, so-called the generalized exponential distribution.

In this paper, we introduce and study several structural properties of a new distribution, referred to as a exponentiated Kumaraswamy-exponential distribution. The properties of the new model are discussed and expressions are derived for the moments, mean deviations, Bonferroni and Lorentz curves. Estimation of the model parameters by the method of maximum likelihood is discussed. The flexibility of this distribution is illustrated in an application to a real data set.

The paper is organized as follows. Section 2 defined the exponentiated Kumaraswamy-exponential distribution and some special sub-models are discussed. Various structural properties which includes moments, mean deviations and Rényi entropy are explored in Section 3. The estimation of the model parameters using the method of maximum likelihood is discussed in Section 4. Finally, in Section 5 an application on a real data set is reported.

The calculations of this paper involve the gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \, e^{-t} dt, \qquad (1.1)$$

the lower incomplete gamma function

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1} \, e^{-t} dt \qquad (1.2)$$

and the upper incomplete gamma function defined by

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} \, e^{-t} dt. \qquad (1.3)$$

2 THE EXPONENTIATED KUMARASWAMY-EXPONEN-TIAL DISTRIBUTION

The cumulative distribution function (CDF) and probability density function (PDF) of the exponential distribution are given by

$$G(x; \alpha) = 1 - \exp(-\alpha x) \qquad (2.1)$$

and

$$g(x; \alpha) = \alpha \exp(-\alpha x), \qquad (2.2)$$

respectively, where $x > 0$ and $\alpha > 0$ is the scale parameter.

[7] introduced a two-parameter distribution, known as Kumaraswamy distribution. Its CDF is given by

$$F(x; \beta, \lambda) = 1 - \left(1 - x^\beta\right)^\lambda \qquad (2.3)$$

where $0 < x < 1$, $\beta > 0$ and $\lambda > 0$. The corresponding PDF for (2.3) is given by

$$f(x; \beta, \lambda) = \beta \lambda x^{\beta-1} \left(1 - x^\beta\right)^{\lambda-1}. \qquad (2.4)$$

The Kumaraswamy distribution has been identified as a viable alternative to beta distribution because they both have the same basic shape properties (unimodal, uniantimodal, increasing, decreaing, monotone or constant). However, the PDF given in Equation 2.4 does not involve any incomplete beta function ratio and it is regarded as being tractable because of its mild algebraic properties. Recently, [8] proposed a generalization of the Kumaraswamy distribution, so-called the exponentiated Kumaraswamy distribution. The CDF and PDF of the exponentiated Kumaraswamy distribution are given by

$$F\left(x; \beta, \lambda, \theta\right) = \left[1 - \left(1 - x^{\beta}\right)^{\lambda}\right]^{\theta} \tag{2.5}$$

and

$$f\left(x; \beta, \lambda, \theta\right) = \beta\lambda\theta x^{\beta-1}\left(1 - x^{\beta}\right)^{\lambda-1}\left[1 - \left(1 - x^{\beta}\right)^{\lambda}\right]^{\theta-1}, \tag{2.6}$$

respectively, where $0 < x < 1$, $\beta > 0$, $\lambda > 0$ and $\theta > 0$ are shape parameters. Let $G(x)$ be the CDF of any random variable X. The CDF of a generalized class of distributions is given by

$$F\left(x; \beta, \lambda, \theta\right) = \left[1 - \left(1 - G(x)^{\beta}\right)^{\lambda}\right]^{\theta}. \tag{2.7}$$

where β, λ and θ are additional shape parameters.

[9] have used the CDF of Dagum distribution in (2.7) to propose the exponentiated Kumaraswamy-Dagum distribution. The generalization (2.7) can be used to propose other distributions based on the exponentiated Kumaraswamy distribution.

Combining (2.1) and (2.7), gives the CDF of the exponentiated Kumaraswamy-exponential (ExpK-E) distribution as:

$$F\left(x; \alpha, \beta, \lambda, \theta\right) = \left\{1 - \left[1 - (1 - \exp\left(-\alpha x\right))^{\beta}\right]^{\lambda}\right\}^{\theta}. \tag{2.8}$$

Differentiating (2.8) with respect to x, and doing the necessary simplifications, gives the PDF as:

$$f\left(x; \alpha, \beta, \lambda, \theta\right) = \alpha\beta\lambda\theta(1 - \exp\left(-\alpha x\right))^{\beta-1}\left[1 - (1 - \exp\left(-\alpha x\right))^{\beta}\right]^{\lambda-1}\exp\left(-\alpha x\right)$$

$$\times \left\{1 - \left[1 - (1 - \exp\left(-\alpha x\right))^{\beta}\right]^{\lambda}\right\}^{\theta-1}. \tag{2.9}$$

where $x > 0$, α, β, λ and θ. α is the scale parameter. β, λ and θ are additional shape parameters.

Figure 1 illustrates some of the possible shapes of the PDF of the ExpK-E distribution for different values of the parameters α, β, λ and θ.

2.1 Sub-models

Sub-models of ExpK-E distribution for selected values of the parameters are presented in this subsection.

1. When $\beta = \lambda = \theta = 1$, the ExpK-E distribution is the exponential distribution with the PDF,

$$f\left(x; \alpha\right) = \alpha\exp\left(-\alpha x\right). \tag{2.10}$$

2. If $\lambda = \theta = 1$, the ExpK-E distribution is the exponentiated exponential (Exp-E) distribution. The PDF is given by

$$f\left(x; \alpha, \beta\right) = \alpha\beta\left(1 - \exp\left(-\alpha x\right)\right)^{\beta-1}\exp\left(-\alpha x\right). \tag{2.11}$$

3. If $\lambda = \beta = 1$, we have the exponentiated exponential distribution with the PDF,

$$f\left(x; \alpha, \theta\right) = \alpha\theta\left(1 - \exp\left(-\alpha x\right)\right)^{\theta-1}\exp\left(-\alpha x\right). \tag{2.12}$$

4. If $\lambda = 1$, we have another exponentiated exponential distribution with parameters α, $\beta\theta$ and PDF is given by

$$f\left(x; \alpha, \beta\theta\right) = \alpha\beta\theta\left(1 - \exp\left(-\alpha x\right)\right)^{\beta\theta-1}\exp\left(-\alpha x\right). \tag{2.13}$$
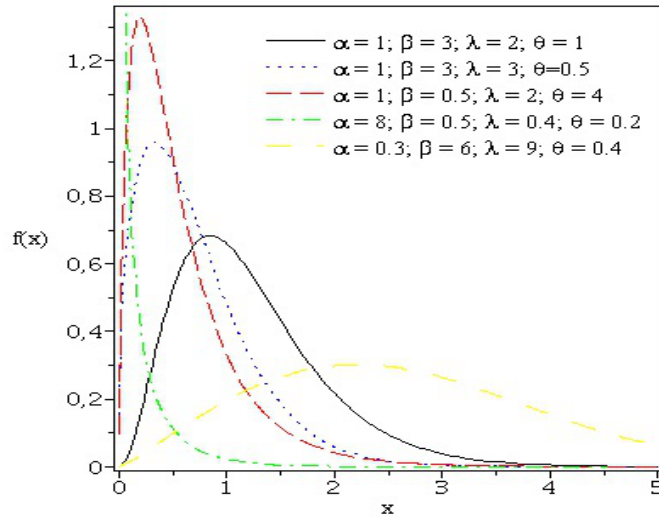
Figure 1: The PDF of the ExpK-E distribution for different values of the parameters

5. When $\theta = 1$, the ExpK-E distribution is the Kumaraswamy exponential distribution introduced by [10] with the FDP given by

$$f(x; \alpha, \beta, \lambda) = \alpha\beta\lambda \left(1 - \exp\left(-\alpha x\right)\right)^{\beta-1} \left[1 - \left(1 - \exp\left(-\alpha x\right)\right)^{\beta}\right]^{\lambda-1} \exp\left(-\alpha x\right). \tag{2.14}$$

6. If $\beta = 1$, we have the exponentiated exponential distribution with parameters $\alpha\lambda$ and θ. The corresponding PDF is

$$f(x; \alpha\lambda, \theta) = \alpha\lambda\theta \left(1 - \exp\left(-\alpha\lambda x\right)\right)^{\theta-1} \exp\left(-\alpha\lambda x\right). \tag{2.15}$$

3 PROPERTIES

3.1 Expansions for the Cumulative and Density Functions

For any real non-integer $\theta > 0$, we have the power series

$$(1 - \omega)^{\theta-1} \quad = \quad \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\theta) \omega^j}{\Gamma(\theta - j) j!} \tag{3.1}$$

where $|\omega| < 1$. Using the power series (3.1) in Equation 2.8, we can write

$$F(x; \alpha, \beta, \lambda, \theta) = \beta\lambda\theta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{j+k+l} jk\Gamma(\theta)\Gamma(\lambda j)\Gamma(\beta k) \exp\left(-\alpha l x\right)}{\Gamma(\theta - j + 1)\Gamma(\lambda j - k + 1)\Gamma(\beta k - l + 1) j! k! l!} \tag{3.2}$$

for β, λ and θ real non-integers. For β integer, the index l in the previous sums stops at βk. If $\lambda > 0$ is an integer, the index k stops at λj. If $\theta > 0$ is an integer, the index j stops at θ.

Now, using again the power series (3.1), we can express (2.9) (for β, λ and θ real non-integers) as

$$f(x; \alpha, \beta, \lambda, \theta) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{j+k+l} \alpha \beta \lambda \theta \Gamma(\theta) \Gamma[\lambda(j+1)] \Gamma[\beta(k+1)] \exp[-\alpha(l+1)x]}{\Gamma(\theta-j)\Gamma[\lambda(j+1)-k]\Gamma[\beta(k+1)-l]\,j!k!l!}. \quad (3.3)$$

The Equation 3.3 reveals that the ExpK-E density function is a linear combination of exponential density functions.

3.2 Hazard Function

For a continuous distribution with PDF $f(x)$ and CDF $F(x)$, the hazard function is defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{P(X < x + \Delta x | X > x)}{\Delta x} = \frac{f(x)}{1-F(x)}. \quad (3.4)$$

The hazard function is an important quantity characterizing life phenomena. For the ExpK-E distribution, the hazard rate function is

$$h(x; \alpha, \beta, \lambda, \theta) = \frac{\alpha \beta \lambda \theta (1 - \exp(-\alpha x))^{\beta-1} \left[1 - (1 - \exp(-\alpha x))^{\beta}\right]^{\lambda-1} \exp(-\alpha x)}{1 - \left\{1 - \left[1 - (1 - \exp(-\alpha x))^{\beta}\right]^{\lambda}\right\}^{\theta}}$$

$$\times \left\{1 - \left[1 - (1 - \exp(-\alpha x))^{\beta}\right]^{\lambda}\right\}^{\theta-1}. \quad (3.5)$$

Figure 2 illustrates the behavior of the hazard function of a ExpK-E distribution for selected values of the parameters α, β, λ and θ.

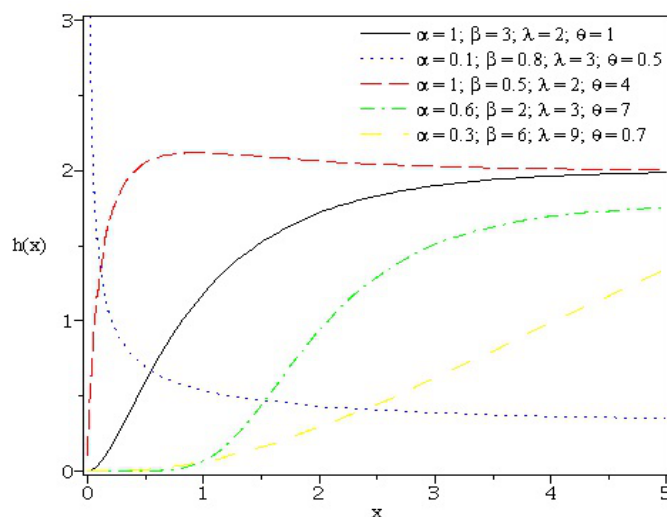

Figure 2: The hazard function of the ExpK-E distribution for different values of the parameters

3.3 Moments

In this subsection we discuss the rth moment for ExpK-E distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis). Therefore, it is customary to derive the moments when a new distribution is proposed.

Using the form in (3.3), we can write

$$
\begin{aligned}
E\left(X^r\right) &= \int_0^\infty x^r f\left(x; \alpha, \beta, \lambda, \theta\right) dx \\
&= \alpha\beta\lambda\theta \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{j+k+l}\Gamma(\theta)\Gamma\left[\lambda(j+1)\right]\Gamma\left[\beta(k+1)\right]}{\Gamma(\theta-j)\Gamma\left[\lambda(j+1)-k\right]\Gamma\left[\beta(k+1)-l\right]j!k!l!} \\
&\quad \times \int_0^\infty x^r exp\left[-\alpha(l+1)x\right] dx.
\end{aligned}
\tag{3.6}
$$

Making the transformation $t = \alpha\left(l+1\right)$ and using the definition of the gamma function (1.1), the rth moment of ExpK-E distribution is given by

$$
E\left(X^r\right) = \alpha\beta\lambda\theta \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{j+k+l}\Gamma(\theta)\Gamma\left[\lambda(j+1)\right]\Gamma\left[\beta(k+1)\right]r!}{\Gamma(\theta-j)\Gamma\left[\lambda(j+1)-k\right]\Gamma\left[\beta(k+1)-l\right]\left[\alpha\left(l+1\right)\right]^{r+1}j!k!l!}.
\tag{3.7}
$$

In particular, the mean for the ExpK-E distribution is given by

$$
E\left(X\right) = \mu = \alpha\beta\lambda\theta \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{j+k+l}\Gamma(\theta)\Gamma\left[\lambda(j+1)\right]\Gamma\left[\beta(k+1)\right]}{\Gamma(\theta-j)\Gamma\left[\lambda(j+1)-k\right]\Gamma\left[\beta(k+1)-l\right]\left[\alpha\left(l+1\right)\right]^2 j!k!l!}.
\tag{3.8}
$$

The moment generating function of the ExpK-E distribution is given by

$$
M\left(t\right) = \alpha\beta\lambda\theta \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{j+k+l}\Gamma(\theta)\Gamma\left[\lambda(j+1)\right]\Gamma\left[\beta(k+1)\right]}{\Gamma(\theta-j)\Gamma\left[\lambda(j+1)-k\right]\Gamma\left[\beta(k+1)-l\right]\left[\alpha\left(l+1\right)-t\right]j!k!l!}
\tag{3.9}
$$

for $t < \alpha(l+1)$. The corresponding characteristic function is

$$
\phi\left(t\right) = \alpha\beta\lambda\theta \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(-1)^{j+k+l}\Gamma(\theta)\Gamma\left[\lambda(j+1)\right]\Gamma\left[\beta(k+1)\right]}{\Gamma(\theta-j)\Gamma\left[\lambda(j+1)-k\right]\Gamma\left[\beta(k+1)-l\right]\left[\alpha\left(l+1\right)-it\right]j!k!l!}
\tag{3.10}
$$

where $i = \sqrt{-1}$.

3.4 Quantiles of the Distribution

The pth quantile, q_p, of the ExpK-E distribution is the real solution of the equation

$$
F(q_p) = p
\tag{3.11}
$$

and is given by:

$$
q_p = -\log\left(1 - \left[1 - \left(1 - p^{1/\theta}\right)^{1/\lambda}\right]^{1/\beta}\right)^{1/\alpha}.
\tag{3.12}
$$

In particular the ExpK-E median is:

$$
q_{0.5} = -\log\left(1 - \left[1 - \left(1 - \left(\frac{1}{2}\right)^{1/\theta}\right)^{1/\lambda}\right]^{1/\beta}\right)^{1/\alpha}.
\tag{3.13}
$$

Using the method of inversion in [11], random numbers from the ExpK-E distribution can be generated with $U \sim U(0,1)$ as the solution of following equation

$$u = \left\{ 1 - \left[1 - (1 - \exp(-\alpha q))^{\beta} \right]^{\lambda} \right\}^{\theta}. \quad (3.14)$$

This yield

$$q = -\log \left(1 - \left[1 - \left(1 - u^{1/\theta} \right)^{1/\lambda} \right]^{1/\beta} \right)^{1/\alpha}. \quad (3.15)$$

Moreover, (3.15) may be used to generate random numbers from the ExpK-E distribution using different initial values for the parameters.

3.5 Mean Deviations

The amount of spread in a population is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median. Let X be a ExpK-E random variable with mean $\mu = E(X)$ and median m.

The mean deviation from the mean can be defined as

$$
\begin{aligned}
\delta_1(X) &= E(|X - \mu|) \\
&= \int_0^{\infty} |x - \mu| f(x; \alpha, \beta, \lambda, \theta)\, dx \\
&= 2\mu F(\mu; \alpha, \beta, \lambda, \theta) - 2\mu + 2 \int_{\mu}^{\infty} x f(x; \alpha, \beta, \lambda, \theta)\, dx \\
&= 2\mu \left\{ 1 - \left[1 - (1 - \exp(-\alpha\mu))^{\beta} \right]^{\lambda} \right\}^{\theta} - 2\mu \\
&\quad + 2\alpha\beta\lambda\theta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\theta)\Gamma[\lambda(j+1)]\Gamma[\beta(k+1)]\Gamma[2,\alpha\mu(l+1)]}{\Gamma(\theta-j)\Gamma[\lambda(j+1)-k]\Gamma[\beta(k+1)-l][\alpha(l+1)]^2 j!k!l!}. \quad (3.16)
\end{aligned}
$$

The mean deviation from the median is, also, defined by

$$
\begin{aligned}
\delta_1(X) &= E(|X - m|) \\
&= \int_0^{\infty} |x - m| f(x; \alpha, \beta, \lambda, \theta)\, dx \\
&= -\mu + 2 \int_m^{\infty} x f(x; \alpha, \beta, \lambda, \theta)\, dx \\
&= -\mu + 2\alpha\beta\lambda\theta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\theta)\Gamma[\lambda(j+1)]\Gamma[\beta(k+1)]\Gamma[2,\alpha m(l+1)]}{\Gamma(\theta-j)\Gamma[\lambda(j+1)-k]\Gamma[\beta(k+1)-l][\alpha(l+1)]^2 j!k!l!}. \quad (3.17)
\end{aligned}
$$

3.6 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined by

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x)\, dx \quad (3.18)$$

and

$$L(p) = \frac{1}{\mu} \int_0^q x f(x)\, dx, \quad (3.19)$$

respectively, where $\mu = E\left(X\right)$ and $q = F^{-1}\left(p\right)$. In the case of EexpK-E distribution, using the results of above paragraphs, we obtain

$$B\left(p\right) \;=\; \frac{\alpha\beta\lambda\theta}{p\mu}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+k+l}\Gamma(\theta)\Gamma\left[\lambda(j+1)\right]\Gamma\left[\beta(k+1)\right]\gamma\left[2,\alpha q\left(l+1\right)\right]}{\Gamma(\theta-j)\Gamma\left[\lambda(j+1)-k\right]\Gamma\left[\beta(k+1)-l\right]\left[\alpha\left(l+1\right)\right]^{2}j!k!l!}$$

(3.20)

and

$$L\left(p\right) \;=\; \frac{\alpha\beta\lambda\theta}{\mu}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+k+l}\Gamma(\theta)\Gamma\left[\lambda(j+1)\right]\Gamma\left[\beta(k+1)\right]\gamma\left[2,\alpha q\left(l+1\right)\right]}{\Gamma(\theta-j)\Gamma\left[\lambda(j+1)-k\right]\Gamma\left[\beta(k+1)-l\right]\left[\alpha\left(l+1\right)\right]^{2}j!k!l!}.$$

(3.21)

3.7 Entropy

An entropy of a random variable X is a measure of variation of the uncertainty. It is an important concept in many fields of science, especially theory of communication, physics and probability. A popular entropy measure is Rényi entropy. If X has the PDF $f(\cdot)$ then Rényi entropy is defined by

$$H_R\left(\nu\right) = \frac{1}{1-\nu}\log\left[\int f^{\nu}\left(x\right)dx\right]$$

(3.22)

where $\nu > 0$ and $\nu \neq 1$. Using (3.22), Rényi entropy of ExpK-E distribution is given by

$$
\begin{aligned}
H_R\left(\nu\right) \;=\;& \frac{\nu}{1-\nu}\left(\log\alpha + \log\beta + \log\lambda + \log\theta\right) \\
+\;& \frac{1}{1-\nu}\log\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+k+l}\Gamma\left[\nu\left(\theta-1\right)+1\right]\Gamma\left[\lambda j + \nu\left(\lambda-1\right)+1\right]}{\left[\alpha\left(l+\nu\right)\right]\Gamma\left[\nu\left(\theta-1\right)+1-j\right]\Gamma\left[j\lambda + \nu\left(\lambda-1\right)+1-k\right]j!k!l!} \\
\times\;& \frac{\Gamma\left[k\beta + \nu\left(\beta-1\right)+1\right]}{\Gamma\left[k\beta + \nu\left(\beta-1\right)+1-l\right]}.
\end{aligned}
$$

(3.23)

3.8 Order Statistics

Order statistics are among the most fundamental tools in non-parametric statistics and inference. They enter in the problems of estimation and hypothesis tests in a variety of ways. Therefore, we now discuss some properties of the order statistics for the ExpK-E distribution. Suppose $X_1,\ X_2,\ldots,X_n$ is a random sample from ExpK-E distribution. Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ denote the corresponding order statistics. From [12], the PDF and CDF of the rth order statistic, say $Y = X_{r:n}$, are given by

$$
\begin{aligned}
f_Y(y) \;=\;& \frac{n!}{(r-1)!(n-r)!}F^{r-1}(y)\left[1-F(y)\right]^{n-r}f(y) \\
=\;& \frac{n!}{(r-1)!(n-r)!}\sum_{l=0}^{n-r}\binom{n-r}{l}(-1)^{l}F^{l+r-1}(y)f(y)
\end{aligned}
$$

(3.24)

and

$$
\begin{aligned}
F_Y(y) \;=\;& \sum_{j=r}^{n}\binom{n}{j}F^{j}(y)\left[1-F(y)\right]^{n-j} \\
=\;& \sum_{j=r}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^{l}F^{j+l}(y)
\end{aligned}
$$

(3.25)

where $f(\cdot)$ and $F(\cdot)$ are the PDF and CDF of the ExpK-E distribution, respectively. It follows from Equations (2.8) and (2.9) that

$$f_Y(y) = \frac{\alpha\beta\lambda\theta\exp\left(-\alpha y\right)n!}{(r-1)!(n-r)!}(1-\exp\left(-\alpha y\right))^{\beta-1}\Big[1-(1-\exp\left(-\alpha y\right))^\beta\Big]^{\lambda-1}$$
$$\times \quad \sum_{l=0}^{n-r}\left(\begin{array}{c}n-r\\l\end{array}\right)(-1)^l\left\{1-\Big[1-(1-\exp\left(-\alpha y\right))^\beta\Big]^\lambda\right\}^{\theta(l+r)-1} \tag{3.26}$$

and

$$F_Y(y)=\sum_{j=r}^{n}\sum_{l=0}^{n-j}\left(\begin{array}{c}n\\j\end{array}\right)\left(\begin{array}{c}n-j\\l\end{array}\right)-1)^l\left\{1-\Big[1-(1-\exp\left(-\alpha y\right))^\beta\Big]^\lambda\right\}^{\theta(j+1)}. \tag{3.27}$$

4 Maximum Likelihood Estimation

Suppose $x_1,\ldots,x_n$ is a random sample of size n from the ExpK-E distribution given by (2.9). The log-likelihood function for the vector of parameters $\Theta=(\alpha,\beta,\lambda,\theta)^T$ can be written as:

$$\log L\left(\Theta\right) = (\beta-1)\sum_{i=1}^{n}\log\left(1-\exp\left(-\alpha x_i\right)\right)+n\log\alpha+n\log\beta+n\log\lambda$$
$$+ \quad (\lambda-1)\sum_{i=1}^{n}\log\Big[1-(1-\exp\left(-\alpha x_i\right))^\beta\Big]+n\log\theta-\alpha\sum_{i=1}^{n}x_i$$
$$+ \quad (\theta-1)\sum_{i=1}^{n}\log\Big[1-\Big(1-(1-\exp\left(-\alpha x_i\right))^\beta\Big)^\lambda\Big]. \tag{4.1}$$

Differentiating the log-likelihood with respect $\alpha,\ \beta,\ \lambda$ and θ, respectively, and setting the result equal to zero, we have

$$\frac{\partial\log L}{\partial\alpha} = \frac{n}{\alpha}+\sum_{i=1}^{n}\frac{\alpha(\beta-1)}{\exp\left(\alpha x_i\right)-1}+\sum_{i=1}^{n}\frac{\alpha\beta(1-\lambda)(1-\exp\left(-\alpha x_i\right))^{\beta-1}}{\Big[1-(1-\exp\left(-\alpha x_i\right))^\beta\Big]\exp\left(\alpha x_i\right)}-\sum_{i=1}^{n}x_i$$
$$+\sum_{i=1}^{n}\frac{\alpha\beta\lambda(\theta-1)(1-\exp\left(-\alpha x_i\right))^{\beta-1}\Big[1-(1-\exp\left(-\alpha x_i\right))^\beta\Big]^{\lambda-1}}{\Big[1-\Big(1-(1-\exp\left(-\alpha x_i\right))^\beta\Big)^\lambda\Big]\exp\left(\alpha x_i\right)}=0, \tag{4.2}$$

$$\frac{\partial\log L}{\partial\beta} = \frac{n}{\beta}+(1-\lambda)\sum_{i=1}^{n}\frac{(1-\exp\left(-\alpha x_i\right))^\beta\log\left(1-\exp\left(-\alpha x_i\right)\right)}{1-(1-\exp\left(-\alpha x_i\right))^\beta}+\sum_{i=1}^{n}\log\left(1-\exp\left(-\alpha x_i\right)\right)$$
$$+ \quad \lambda(\theta-1)\sum_{i=1}^{n}\frac{\Big[1-(1-\exp\left(-\alpha x_i\right))^\beta\Big]^{\lambda-1}(1-\exp\left(-\alpha x_i\right))^\beta}{1-\Big[1-(1-\exp\left(-\alpha x_i\right))^\beta\Big]^\lambda}$$
$$\times \quad \log\left(1-\exp\left(-\alpha x_i\right)\right)=0, \tag{4.3}$$

$$\frac{\partial\log L}{\partial\lambda} = \frac{n}{\lambda}+\sum_{i=1}^{n}\log\Big[1-(1-\exp\left(-\alpha x_i\right))^\beta\Big]+(\theta-1)\sum_{i=1}^{n}\frac{\Big[1-(1-\exp\left(-\alpha x_i\right))^\beta\Big]^\lambda}{1-\Big[1-(1-\exp\left(-\alpha x_i\right))^\beta\Big]^\lambda}$$
$$\times \quad \log\Big(1-(1-\exp\left(-\alpha x_i\right))^\beta\Big)=0, \tag{4.4}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log \left[1 - \left(1 - (1 - \exp(-\alpha x_i))^\beta \right)^\lambda \right] = 0. \tag{4.5}$$

The maximum likelihood estimator (MLE) $\widehat{\Theta}$ is obtained by solving Equations 4.2−4.5. To solve (4.2) through (4.5), it is usually more convenient to use nonlinear optimization algorithms such as quasi−Newton algorithm to numerically maximize the log-likelihood function. In order to compute the standard errors and asymptotic confidence intervals the usual large sample approximation is used, in which the maximum likelihood estimators can be treated as being approximately multivariate normal. Hence as $n \to \infty$, the asymptotic distribution of the MLE is given by,

$$\begin{pmatrix} \widehat{\alpha} \\ \widehat{\beta} \\ \widehat{\lambda} \\ \widehat{\theta} \end{pmatrix} = N \left[\begin{pmatrix} \alpha \\ \beta \\ \lambda \\ \theta \end{pmatrix}, \begin{pmatrix} \widehat{V}_{11} & \widehat{V}_{12} & \widehat{V}_{13} & \widehat{V}_{14} \\ \widehat{V}_{21} & \widehat{V}_{22} & \widehat{V}_{23} & \widehat{V}_{24} \\ \widehat{V}_{31} & \widehat{V}_{32} & \widehat{V}_{33} & \widehat{V}_{34} \\ \widehat{V}_{41} & \widehat{V}_{42} & \widehat{V}_{43} & \widehat{V}_{44} \end{pmatrix} \right] \tag{4.6}$$

where $\widehat{V}_{ij} = V_{ij}|_{\Theta=\widehat{\Theta}}$ and

$$\begin{pmatrix} \widehat{V}_{11} & \widehat{V}_{12} & \widehat{V}_{13} & \widehat{V}_{14} \\ \widehat{V}_{21} & \widehat{V}_{22} & \widehat{V}_{23} & \widehat{V}_{24} \\ \widehat{V}_{31} & \widehat{V}_{32} & \widehat{V}_{33} & \widehat{V}_{34} \\ \widehat{V}_{41} & \widehat{V}_{42} & \widehat{V}_{43} & \widehat{V}_{44} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}^{-1} \tag{4.7}$$

is the approximate variance covariance matrix with elements obtained from

$$A_{11} = -\frac{\partial^2 \log L}{\partial \alpha^2}, \quad A_{12} = -\frac{\partial^2 \log L}{\partial \alpha \partial \beta}, \quad A_{13} = -\frac{\partial^2 \log L}{\partial \alpha \partial \lambda}, \quad A_{14} = -\frac{\partial^2 \log L}{\partial \alpha \partial \theta},$$

$$A_{22} = -\frac{\partial^2 \log L}{\partial \beta^2}, \quad A_{23} = -\frac{\partial^2 \log L}{\partial \beta \partial \lambda}, \quad A_{24} = -\frac{\partial^2 \log L}{\partial \beta \partial \theta},$$

$$A_{33} = -\frac{\partial^2 \log L}{\partial \lambda^2}, \quad A_{34} = -\frac{\partial^2 \log L}{\partial \lambda \partial \theta},$$

$$A_{44} = -\frac{\partial^2 \log L}{\partial \theta^2}. \tag{4.8}$$

Approximate $100(1 - \gamma)\%$ confidence intervals two sided confidence intervals for α, β, λ and θ are, respectively, given by

$$\widehat{\alpha} \pm z_{\gamma/2}\sqrt{\widehat{V}_{11}}, \quad \widehat{\beta} \pm z_{\gamma/2}\sqrt{\widehat{V}_{22}}, \quad \widehat{\lambda} \pm z_{\gamma/2}\sqrt{\widehat{V}_{33}}, \quad \widehat{\theta} \pm z_{\gamma/2}\sqrt{\widehat{V}_{44}} \tag{4.9}$$

where z_γ is the upper γ^{th} percentile of the standard normal distribution.

5 Application

In this section we will study one real data set to illustrate the usefulness of the ExpK-E distribution for modeling reliability data. We will make comparison of the results with the exponential (Exp), inverse exponential (IExp) and Weibull distributions.

We consider the widely used data from [13]. The data represent the survival times of guinea pigs injected with different doses of tubercle bacilli.

The maximum likelihood estimates, the corresponding values of log-likelihood and the Akaike Information Criterion (AIC) values for the fitted distributions are reported in Table 1. The results show that the ExpK-E distribution provides a significantly better fit than the other models.

Plots of the estimated PDF of the exponential, ExpK-E, IE and Weibull models fitted to these data set are given in Figure 3. The figure indicate that the ExpK-E distribution is superior to the other distributions in terms of model fitting.
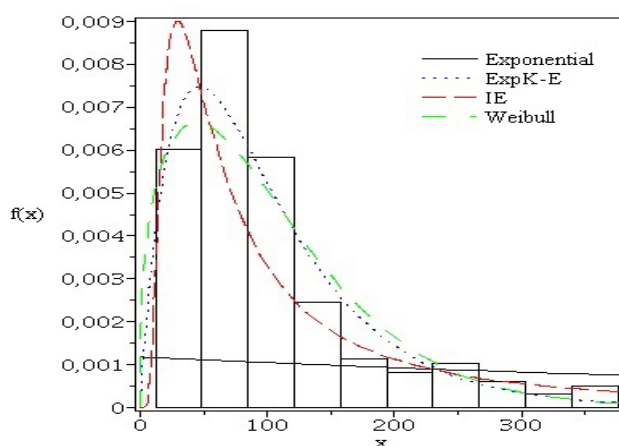

Figure 3: Histogram and estimated densities.

Table 1. The maximum likelihood estimates and AIC of the models

Model	Maximum likelihood estimates	log-likelihood	AIC
Exp	$\alpha = 0.001,\ \beta = 1,\ \lambda = 1,\ \theta = 1$	-493.887	989.775
ExpK-E	$\alpha = 0.015,\ \beta = 2.010,\ \lambda = 1.001,\ \theta = 1.010$	-393.639	795.279
IExp	$\alpha = 60.098$	-402.502	807.003
Weibull	$\alpha = 0.009,\ \beta = 1.392$	-397.148	798.300

6   CONCLUSION

We proposed a new distribution, named the exponentiated Kumaraswamy-exponential distribution which extends the exponential distribution. Several properties of the new distribution were investigated. The estimation of parameters by the method of moments and the maximum likelihood have been discussed. An application of the exponentiated Kumaraswamy-exponential distribution to real data show that the new distribution can be used quite effectively to provide better fits than the exponential, inverse exponential and Weibull distributions.

COMPETING INTERESTS

The authors declare that no competing interests exist.

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